

KdV-like solitary waves in two-dimensional FPU-lattices

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Abstract

We prove the existence of solitary waves in the KdV limit of two-dimensional FPU-type lattices using asymptotic analysis of nonlinear and singularly perturbed integral equations. In particular, we generalize the existing results by Friesecke and Matthies since we allow for arbitrary propagation directions and non-unidirectional wave profiles.

Keywords: *two-dimensional FPU-lattices, KdV limit of lattice waves, asymptotic analysis of singularly perturbed integral equations*

MSC (2010): 37K60, 37K40, 74H10

Contents

1	Introduction	1
1.1	Setting of the problem	2
1.2	The proof strategy and the main result	4
2	Abstract existence result	6
2.1	Preliminaries	6
2.2	Reformulation as fixed point problem and basic assumptions	8
2.3	Further properties of \mathcal{B}_ϵ , \mathcal{M}_ϵ and \mathcal{L}_ϵ	13
2.4	Inverting the Operator \mathcal{L}_ϵ	16
2.5	Fixed-point Argument	20
3	Applications to different lattices	21
3.1	Square lattice	22
3.2	Diamond lattice	24
3.3	Triangle lattice	25

1 Introduction

Since the pioneering work [1] a number of further research papers have been devoted to the understanding of waves arising in 1D FPU-lattices, of which the only nontrivial completely integrable type is the so-called Toda lattice, that admits explicit traveling wave solutions in terms of Jacobian elliptic functions, see for instance [4]. Among these works [2] was the first to reveal a connection between solitary waves with long-wave-lengths and small amplitudes and the Kortewegde Vries (KdV) equation. By diminishing the lattice spacing the authors were able to pass to a continuum described

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by a KdV-equation, which is a completely integrable PDE. However, since their analysis was purely formal, it didn't lead to a rigorous proof. The existence of KdV-like traveling waves in FPU chains was justified in a rigorous way much later, in the first of a series of papers [5], [6], [7] and [8], of which the other three deal with the stability theory. The proof was based on the introduction of a new framework for passing to the continuum and concerned only lattices with interactions between nearest neighbours, see also [4, 15, 13] for related results. The existence result [5] is extended by Herrmann and Mikikits-Leitner [10] to 1D lattices with nonlocal interactions, where the original argument by a careful analysis of poles of the Fourier multiplier in the complex plane is replaced by direct estimates on the real line. We mention here that for considerably general initial data with wavelength L it is shown in [14] that FPU solutions can be approximately expressed as the sum of two uncoupled counter-propagating KdV solutions on timescales $\sim L^3$. Both the ansatz employed in [5] and [10] can be viewed as special cases thereof.

Existence results on various types of waves in FPU-chains can be found in the literature. G. Iooss [21] provides a precise description of all subsonic and supersonic waves with small amplitudes via the center manifold reduction. In [19] the existence of solitary waves with large amplitudes was proved for the first time by applying the concentration-compactness principle, see also [16] for similar results under slightly different conditions. Unimodal solitary waves are obtained in [9] by studying a convex potential functional within a certain cone, where many different conditions for the physical potentials are collected. In all these results the wave speed is in principle unknown. However, there is a way of finding both periodic and solitary waves with any prescribed wave speed above a critical value [20], which employs a special version of mountain pass theorem. In [22] the existence of solitary waves is proved by approximation with periodic solutions, which are obtained by means of the standard mountain pass argument. Aside from homoclinic waves, heteroclinic solutions can also be found in FPU-chains: in [18] it is shown that heteroclinic fronts exist in FPU-chains for certain convex potentials.

The investigations mentioned above concern exclusively 1D FPU-lattices. In the practice of physics, two or more dimensional FPU-lattices are more relevant, since they provide simplified models for crystals and solids such as metals, etc. However, to the best of our knowledge, no existence result for KdV-like waves in such FPU-lattices has been obtained except for a degenerate case [3], where it is assumed that the wave (both the distance and velocity profiles) is unidirectional and longitudinal moving along the horizontal direction. This assumption reduces the original two-dimensional system to a one-dimensional one and the geometric non-linearity arising from physical linearity due to the introduction of diagonal springs enables one to apply results from [5] to obtain the existence of KdV-like solitary waves and also its asymptotic profile shape.

In the present paper we show an existence result for 2D FPU-lattices in a general framework, which covers different lattice geometries and allows for arbitrary propagating directions. Our asymptotic approach adapts some arguments developed in [10], but is more sophisticated due to the two-dimensional setting.

The discussion of 2D FPU-lattices in the present paper hints at similar results for 3D FPU-lattices, which have to be left for further investigations. Besides, the study of stability of the resulting solutions requires ideas lying beyond the scope of the present paper and will be here completely omitted.

1.1 Setting of the problem

In this subsection we consider as an example the square lattice displayed in the left panel of Figure 1, which leads to the general system of equations. We assume that the lattice is indexed by $(i, j) \in \mathbb{Z}^2$ and the position of the $(i, j)^{\text{th}}$ particle is given by

$$\begin{pmatrix} r_* i \\ r_* j \end{pmatrix} + q_{ij}(t),$$

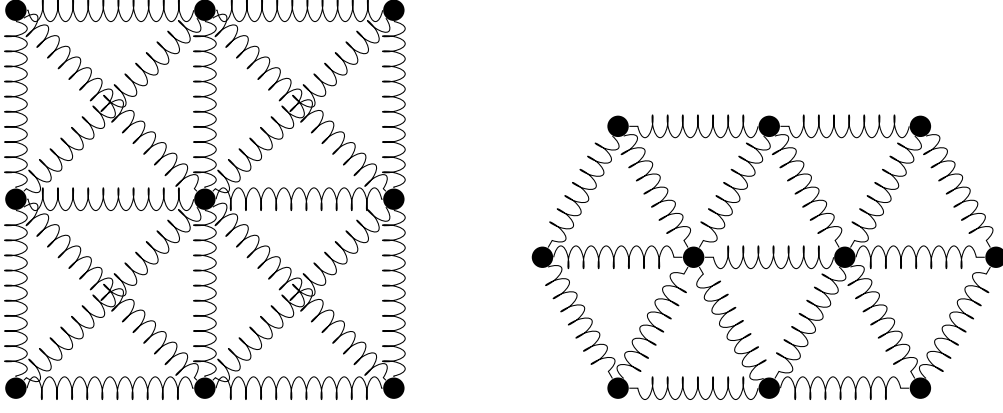


Figure 1: *Left panel:* four periods are shown of the square lattice. The vertical resp. horizontal potentials are denoted by V_1 and the diagonal ones by V_2 . *Right panel:* triangle lattice, where the interaction potentials are the same and denoted by V . The essential difference of the two lattices lies in the symmetry: their symmetry groups \mathcal{D}_4 and \mathcal{D}_3 are non-isomorphic.

where $r_* > 0$ is a reference lattice parameter and $q_{ij}(t) = (q_{ij,1}(t), q_{ij,2}(t))^T \in \mathbb{R}^2$ represents the displacement of the $(i, j)^{\text{th}}$ particle at time t . By summing up the forces exerted on a given particle by its eight neighbors and applying the second Newton's law we obtain

$$\begin{aligned} \ddot{q}_{ij} = & ((\nabla\phi_1)(q_{i+1j} - q_{ij}) - (\nabla\phi_1)(q_{ij} - q_{i-1j})) \\ & + ((\nabla\phi_2)(q_{ij+1} - q_{ij}) - (\nabla\phi_2)(q_{ij} - q_{ij-1})) \\ & + ((\nabla\phi_3)(q_{i+1j+1} - q_{ij}) - (\nabla\phi_3)(q_{ij} - q_{i-1j-1})) \\ & + ((\nabla\phi_4)(q_{i+1j-1} - q_{ij}) - (\nabla\phi_4)(q_{ij} - q_{i-1j+1})), \end{aligned} \quad (1)$$

where

$$\begin{aligned} \phi_1(x_1, x_2) &= V_1(\sqrt{(x_1 + r_*)^2 + x_2^2}), \quad \phi_2(x_1, x_2) = V_1(\sqrt{x_1^2 + (x_2 + r_*)^2}), \\ \phi_3(x_1, x_2) &= V_2(\sqrt{(x_1 + r_*)^2 + (x_2 + r_*)^2}), \quad \phi_4(x_1, x_2) = V_2(\sqrt{(x_1 + r_*)^2 + (x_2 - r_*)^2}). \end{aligned}$$

are the four effective potentials corresponding to the horizontal, vertical, diagonal springs respectively. In this paper we are exclusively interested in traveling wave solutions of the following type

$$q_{ij}(t) = \epsilon U_\epsilon \left(\epsilon(\kappa_1 i + \kappa_2 j - c_\epsilon t) \right) \in \mathbb{R}^2 \quad (2)$$

where $\kappa := (\kappa_1, \kappa_2)^T$ is the wave vector prescribing the propagating direction of the wave and $c_\epsilon > 0$ the wave speed that depends on the parameter $\epsilon > 0$. By a suitable scaling we can assume that κ is normalized with respect to its length. Denote by α the angle between κ and the positive axis. Then we can write $\kappa = (\cos(\alpha), \sin(\alpha))^T$.

Substituting ansatz (2) into (1), we arrive at a system of difference-differential equations for \tilde{q} with one forward and one backward delay for each $m = 1, \dots, M$:

$$\epsilon^3 c_\epsilon^2 \tilde{q}_\epsilon'' = \sum_{m=1}^M k_m \epsilon \delta_{-k_m \epsilon} \tilde{F}^m(k_m \epsilon^2 \delta_{+k_m \epsilon} \tilde{q}_{\epsilon,1}, k_m \epsilon^2 \delta_{+k_m \epsilon} \tilde{q}_{\epsilon,2}), \quad (3)$$

where $k_1 = \kappa_1$, $k_2 = \kappa_2$, $k_3 = \kappa_1 + \kappa_2$, $k_4 = \kappa_1 - \kappa_2$ and $\tilde{F}^m = \nabla\phi_m$ for $m = 1, \dots, M$. Moreover, the right hand side of this equation is formulated in terms of difference operators

$$\delta_{+k_m \epsilon} Y := \frac{Y(\cdot + k_m \epsilon) - Y(\cdot)}{k_m \epsilon}, \quad \delta_{-k_m \epsilon} Y := \frac{Y(\cdot - k_m \epsilon) - Y(\cdot)}{k_m \epsilon}$$

for $Y \in \mathbf{L}^2(\mathbb{R})$, which lead in a natural way to the integral operator acting on the velocity profiles $W_\epsilon := \tilde{q}'_\epsilon$: the difference of \tilde{q}_ϵ is easily obtained by integrating its derivative. It is also straightforward to obtain the corresponding equation for the triangle lattice displayed in Figure 1 by choosing the suitable effective potentials and projecting the wave vector on the axes of symmetry, see section 3 for the details.

In the general case, we shall always assume that \tilde{F}_i^m , which take over the role of the partial derivatives of effective potentials in the square and triangle lattices, have the form

$$\tilde{F}_i^m(x_1, x_2) = \alpha_{i,1}^m x_1 + \alpha_{i,2}^m x_2 + \frac{1}{2}(\beta_{i,11}^m x_1^2 + 2\beta_{i,12}^m x_1 x_2 + \beta_{i,22}^m x_2^2) + \Psi_i^m(x_1, x_2) \quad (4)$$

for all $i = 1, 2$ and $m = 1, \dots, M$, where $\Psi_i^m(x_1, x_2)$ represent the higher order terms. For small $\Psi_i^m(x_1, x_2)$ the existence of KdV-like waves only depends on the linear and quadratic coefficients. We shall first prove the existence result for (3) with arbitrary M , k_m and \tilde{F}^m and then consider special lattices, where these quantities take on concrete forms.

1.2 The proof strategy and the main result

Following Herrmann and Mikikits-Leitner [10] we consider the velocity profile W_ϵ and decompose it as the sum of a limit profile W_0 and an $O(\epsilon^2)$ -corrector:

$$W_\epsilon := W_0 + \epsilon^2 V_\epsilon. \quad (5)$$

The square of the wave speed $\sigma_\epsilon := c_\epsilon^2$ is accordingly broken up as a constant part σ_0 and a remainder term ϵ^2 :

$$\sigma_\epsilon := \sigma_0 + \epsilon^2, \quad (6)$$

where $\sigma_0 > 0$ is completely determined by the linear coefficients $\alpha_{i,j}^m$, see formula (15). Applying (6) and summarizing linear terms with respect to W_ϵ in (12) on the left hand side and the remainder terms on the right, we arrive at the equation

$$\mathcal{B}_\epsilon[W_\epsilon] = \mathcal{Q}_\epsilon[W_\epsilon] + \epsilon^2 \mathcal{P}_\epsilon[W_\epsilon], \quad (7)$$

where \mathcal{B}_ϵ is a linear integral operator introduced in (16), \mathcal{Q}_ϵ depends quadratically on W_ϵ and \mathcal{P}_ϵ stems from the higher order terms Ψ_i^m .

The key observation for the considerations in this paper is that the formal limit equation

$$B_0[W_0] = \mathcal{Q}_0[W_0] \quad (8)$$

is equivalent to

$$W_{0,1} = W_*, \quad W_{0,2} = \lambda W_*, \quad (9)$$

where W_* is the unique even and homoclinic solution to the ODE

$$W_*'' = d_1 W_* - d_2 W_*^2. \quad (10)$$

Here d_1 and d_2 are positive constants depending on the linear and quadratic coefficients $\alpha_{i,j}^m$ and $\beta_{i,jk}^m$ and the scalar quotient $\lambda = W_{0,2}/W_{0,1}$ can be computed explicitly, see (16). Notice that W_* defines via $w(t, \xi) := W_*(\xi - t)$ the solitary wave for the KdV equation

$$d_1 \partial_t w + d_2 \partial_\xi w^2 + \partial_\xi^3 w = 0.$$

At the next level, we will turn (10) into a fixed-point equation with respect to the corrector V_ϵ by inserting (5) and rearranging the terms in such a way, that the linear terms with respect to

V_ϵ stand on the left hand side, which yield automatically a linear operator \mathcal{L}_ϵ . After the uniform inversion of \mathcal{L}_ϵ on $(\mathcal{L}_{\text{even}}^2(\mathbb{R}))^2$, see Theorem 17, we obtain

$$V_\epsilon = \mathcal{F}_\epsilon[V_\epsilon].$$

The contraction property of \mathcal{F}_ϵ , see (23), will be shown in a sufficiently large ball in $(\mathcal{L}_{\text{even}}^2(\mathbb{R}))^2$. Then the Banach fixed-point theorem provides us with a solution, which is at the same time unique in the ball chosen. We note that the uniqueness of solutions in general is a notoriously difficult unsolved problem in the study of lattice waves.

One finds similarities in the technical details parallel with [10]. However, the arguments are always adapted to the two-dimensional situation: for instance, the counterpart of \mathcal{B}_ϵ is no more symmetric and its inversion involves the study of a matrix of operators instead of a single operator on $\mathcal{L}^2(\mathbb{R})$.

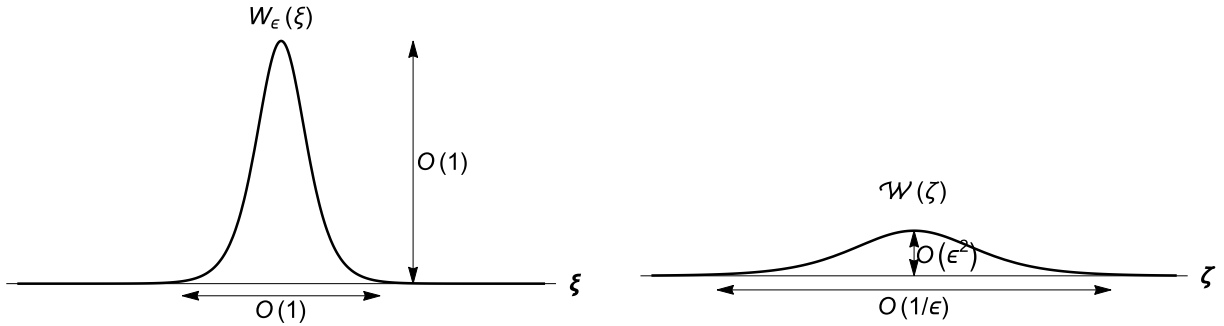


Figure 2: *Left panel:* scaled profile $W_\epsilon(\xi)$; *Right panel:* unscaled profile $W(\zeta)$. The unscaled profile is obtained from the scaled one by stretching in the argument space by $\frac{1}{\epsilon}$ and pressing the amplitude by ϵ^2 .

Our main result is summarized in the following theorem, where Assumptions 5, 5, 6 and 9 restricting the coefficients of \tilde{F}_i^m are to be specified later on. This is at first an abstract result, which is independent of any lattice. In special lattices one has to verify the required assumptions in concrete forms as we shall see in the final section.

Theorem 1 *Under Assumptions 4, 5, 6 and 9 there exists a family (σ_0, W_0) , where σ_0 is a positive constant and $W_0 \in (\mathcal{L}_{\text{even}}^2(\mathbb{R}))^2$ is the KdV traveling wave (as defined above), such that for sufficiently small $\epsilon > 0$ and wave speed $c_\epsilon = \sqrt{\sigma_0 + \epsilon^2}$, equation (3) has a solution, whose velocity profile $W_\epsilon \in (\mathcal{L}_{\text{even}}^2(\mathbb{R}))^2$ satisfies $\|W_\epsilon - W_0\|_2 \leq C\epsilon^2$.*

This paper is organized as follows: in section 2.1 the original problem is reformulated as an eigenvalue problem in W_ϵ , then in section 2.2 as an fixed point equation with respect to the corrector V_ϵ , thereby motivating several assumptions that turn out to guarantee the existence of a solution. Afterwards in section 2.3 the new equation is examined term by term and the properties of the ensuing operators are discussed for later use. We prove in section 2.4 the key asymptotic result, i.e., we show that the operator \mathcal{L}_ϵ , which stems from the linearization of (7), is uniformly invertible on $(\mathcal{L}_{\text{even}}^2(\mathbb{R}))^2$. Then in section 2.5 the contraction property of \mathcal{F}_ϵ is verified to conclude the proof. Finally in section 3 three different lattices are discussed as applications of our main result.

2 Abstract existence result

2.1 Preliminaries

For technical convenience we recast (3) in terms of integral operators. For an arbitrary constant $\eta > 0$, we define the integral operator \mathcal{A}_η by

$$(\mathcal{A}_\eta Y)(\xi) := \frac{1}{\eta} \int_{\xi - \frac{\eta}{2}}^{\xi + \frac{\eta}{2}} Y(\tilde{\xi}) d\tilde{\xi},$$

for $Y \in \mathbf{L}^2(\mathbb{R})$. This operator corresponds to $\text{sinc}(\eta z/2)$ in the Fourier space and can be formally expanded as

$$\mathcal{A}_\eta = \text{id} + \frac{\eta^2}{24} \partial^2 + O(\eta^4). \quad (11)$$

In other words, \mathcal{A}_η is a singular perturbation of the identity due to the remainder terms with higher derivatives and this complicates the asymptotic analysis. Fortunately, the derivation of the decisive properties of the auxiliary operator \mathcal{B}_ϵ , see (16), under certain generic conditions only involves the inversion of the symbol function

$$1 + \sum_{m=1}^M \gamma_m \frac{1 - \text{sinc}(k_m z/2)^2}{\epsilon^2}$$

in the Fourier space which corresponds to the inverse operator of the type

$$\left(\text{id} + \sum_{m=1}^M \gamma_m \frac{\text{id} - \mathcal{A}_{k_m}^2}{\epsilon^2} \right)^{-1}$$

The asymptotic properties of such an operator are well understood in [10], see Lemma 8 and (24) of the present paper and compare Lemma 6 in [10].

As mentioned in the last section, the operator \mathcal{A}_ϵ allows our problem to be reformulated as one concerning the velocity profile $W_\epsilon := \tilde{q}'_\epsilon$ and this leads to a new framework for solving (3).

Lemma 2 *Suppose $W_\epsilon := \tilde{q}'_\epsilon \in (\mathbf{L}^2(\mathbb{R}))^2$. Then the traveling wave equation (3) is equivalent to the nonlinear eigenvalue problem*

$$\epsilon^2 \sigma_\epsilon W_\epsilon = \sum_{m=1}^M k_m \mathcal{A}_{k_m \epsilon} \tilde{F}^m (\epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_{\epsilon,1}, \epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_{\epsilon,2}) \quad (12)$$

with unknown eigenfunction W_ϵ .

Proof: Suppose (12) is given. Let $\tilde{q}_\epsilon(\zeta) := \int_{\zeta_0}^\zeta W_\epsilon(\tilde{\zeta}) d\tilde{\zeta}$, where ζ_0 is an arbitrary constant. Then we have $\mathcal{A}_\eta W_\epsilon = (\delta_{+\eta} \tilde{q}_\epsilon)(\cdot - \frac{\eta}{2})$ by definition. Note that $(\mathcal{A}_\eta Y)' = (\delta_{-\eta} Y)(\cdot + \frac{\eta}{2})$ holds for $Y \in \mathbf{L}^2(\mathbb{R})$. Differentiating both sides of (12) we have on the left hand side $\epsilon^2 \sigma_\epsilon \tilde{q}_\epsilon''$ and on the right hand side $\pm \frac{k_m \epsilon}{2}$ cancel each other in the argument, so that we obtain the right hand side of (3). Integrating (3) we obtain immediately (12) plus a constant vector $\in \mathbb{R}^2$. Since $W_\epsilon \in (\mathbf{L}^2(\mathbb{R}))^2$ and $\tilde{F}^m((0,0)) = (0,0)$ for all m , the constant is zero. \square

We will heavily rely on the properties of \mathcal{A}_η that we now summarize in the following lemma. Notice that (13) justifies in a rigorous way the formal expansion (11) of \mathcal{A}_η .

Lemma 3 *For each $\eta > 0$, the integral operator \mathcal{A}_η has the following properties:*

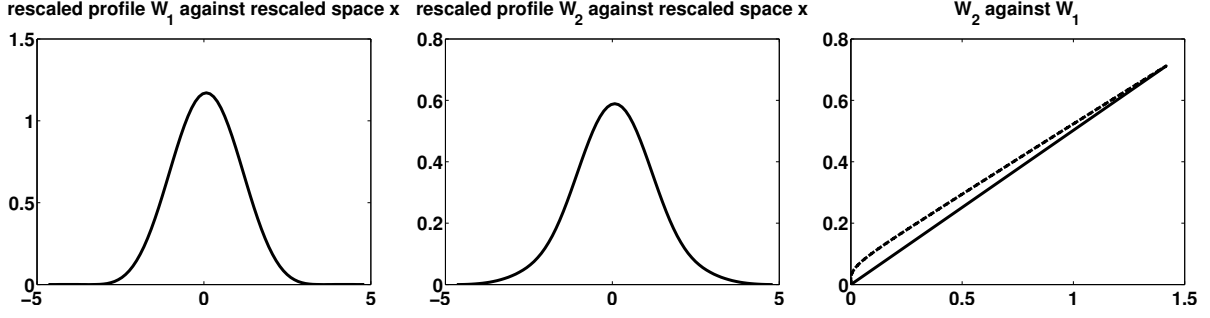


Figure 3: *Left panel:* $W_{\epsilon,1}(\xi)$; *Middle panel:* $W_{\epsilon,2}(\xi)$; *Right panel:* Comparison of $W_{\epsilon,2}(\xi)$ and $W_{\epsilon,1}(\xi)$. The scaled velocity profile W_ϵ is numerically computed for small $\epsilon > 0$ and angle $\alpha = \frac{\pi}{8}$. In the right panel it is clearly seen that the two components of W_ϵ are not proportional, which means that W_ϵ is not unidirectional and our problem cannot be reduced to a one-dimensional one for general angles α .

1. For any $W \in \mathbf{L}^2(\mathbb{R})$, we have $\mathcal{A}_\eta \in \mathbf{L}^2(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ with

$$\|\mathcal{A}_\eta W\|_\infty \leq \eta^{-1/2} \|W\|_2, \quad \|\mathcal{A}_\eta W\|_2 \leq \|W\|_2.$$

Moreover, \mathcal{A}_η admits a weak derivative with $\|(\mathcal{A}_\eta W)'\|_2 \leq \eta^{-1/2} \|W\|_2$.

2. For any $W \in \mathbf{L}^\infty(\mathbb{R})$, we have $\|\mathcal{A}_\eta W\|_\infty \leq \|W\|_\infty$.
3. \mathcal{A}_η respects the even-odd parity, the non-negativity and the unimodality of functions. The latter means monotonicity for both negative and positive arguments.
4. \mathcal{A}_η diagonalizes in Fourier space and corresponds to the symbol function

$$a_\eta(z) = \text{sinc}(\eta z/2)$$

with $\text{sinc}(z) := \sin(z)/z$.

5. \mathcal{A}_η is self-adjoint in the \mathbf{L}^2 -sense.
6. The operators \mathcal{A}_{η_1} and \mathcal{A}_{η_2} commute with each other for any $\eta_1, \eta_2 > 0$.
7. There exists a constant $C > 0$, which does not depend on η , such that the estimates

$$\|\mathcal{A}_\eta W - W\|_2 \leq C\eta^2 \|W''\|_2, \quad \|\mathcal{A}_\eta W - W\|_\infty \leq C\eta^2 \|W''\|_\infty$$

and

$$\|\mathcal{A}_\eta W - W - \frac{\eta^2}{24} W''\|_2 \leq C\eta^4 \|W''''\|_2, \quad \|\mathcal{A}_\eta W - W - \frac{\eta^2}{24} W''\|_\infty \leq C\eta^4 \|W''''\|_\infty$$

hold for any sufficiently regular W . In particular, we have

$$\mathcal{A}_\eta W \rightarrow W \quad \text{strongly in } \mathbf{L}^2(\mathbb{R}) \quad (13)$$

for any $W \in \mathbf{L}^2(\mathbb{R})$.

Proof: All the assertions follow from standard arguments, see [9], [16] and [17] for the details. \square

2.2 Reformulation as fixed point problem and basic assumptions

The purpose of the present section is twofold: to rewrite system (12) as a fixed point equation with respect to V_ϵ and to motivate the assumptions that finally turn out to suffice to guarantee the existence of KdV-like solutions.

Since for traveling waves the functions \tilde{F}_i^m represent the partial derivatives of the effective potentials, see for instance (1), a natural condition can be formulated as follows.

Assumption 4 *The coefficients of \tilde{F}_i^m , see (4), satisfy*

$$\alpha_{1,2}^m = \alpha_{2,1}^m, \quad \beta_{1,22}^m = \beta_{2,12}^m, \quad \beta_{1,12}^m = \beta_{2,11}^m,$$

for all $m = 1, \dots, M$

As already explained in the introduction, our strategy is to collect the linear and the quadratic terms from (12) into two operators \mathcal{B}_ϵ and \mathcal{Q}_ϵ , respectively, as in (7). A first natural choice for the linear part would be

$$\mathcal{B}_\epsilon^{\text{can}} := \frac{1}{\epsilon^2} \begin{pmatrix} \sigma_\epsilon - \sum_{m=1}^M k_m^2 \alpha_{1,1}^m \mathcal{A}_{k_m \epsilon}^2 & - \sum_{m=1}^M k_m^2 \alpha_{1,2}^m \mathcal{A}_{k_m \epsilon}^2 \\ - \sum_{m=1}^M k_m^2 \alpha_{2,1}^m \mathcal{A}_{k_m \epsilon}^2 & \sigma_\epsilon - \sum_{m=1}^M k_m^2 \alpha_{2,2}^m \mathcal{A}_{k_m \epsilon}^2 \end{pmatrix},$$

but this operator does not converge as $\epsilon \rightarrow 0$. In fact, thanks to (6) the dominant part is given by

$$\mathcal{B}_\epsilon^{\text{sing}} := \frac{1}{\epsilon^2} \begin{pmatrix} \sigma_0 - c_1 & -c_2 \\ -c_2 & \sigma_0 - c_3 \end{pmatrix}$$

with

$$c_1 := \sum_{m=1}^M k_m^2 \alpha_{1,1}^m, \quad c_2 := \sum_{m=1}^M k_m^2 \alpha_{1,2}^m = \sum_{m=1}^M k_m^2 \alpha_{2,1}^m, \quad c_3 := \sum_{m=1}^M k_m^2 \alpha_{2,2}^m$$

and diverges as $\epsilon \rightarrow 0$. In what follows we therefore define the operator \mathcal{B}_ϵ in a slightly different way and choose σ_0 such that $\mathcal{B}_\epsilon^{\text{sing}}$ has a nontrivial kernel. This reads

$$(c_1 - \sigma_0)(c_3 - \sigma_0) = c_2^2 \tag{14}$$

and provides two possible solution branches. However, it turns out that only the larger value of σ_0 is admissible for our asymptotic analysis, see the remarks at the beginning of section 3, and thus we set

$$\sigma_0 = \frac{(c_1 + c_3) + \sqrt{(c_1 - c_3)^2 + c_2^2}}{2}. \tag{15}$$

Using this particular choice of σ_0 we now define

$$\mathcal{B}_\epsilon := \mathcal{T}_\epsilon \cdot \mathcal{B}_\epsilon^{\text{can}}$$

with

$$\mathcal{T}_\epsilon := \begin{pmatrix} 1 & \lambda \\ 0 & \epsilon^2 \end{pmatrix}, \quad \lambda := \frac{c_2}{\sigma_0 - c_3} = \frac{\sigma_0 - c_1}{c_2}, \tag{16}$$

such that the corresponding singular part is given by

$$\mathcal{T}_\epsilon \cdot \mathcal{B}_\epsilon^{\text{sing}} = \begin{pmatrix} 0 & 0 \\ -\tilde{c}_2 & \sigma_0 - c_3 \end{pmatrix}.$$

Using this definition of \mathcal{B}_ϵ , the nonlinear integral equation (12) can be reformulated as (7), where the quadratic and higher terms correspond to operators \mathcal{Q}_ϵ and \mathcal{P}_ϵ with components

$$\begin{aligned} (\mathcal{Q}_\epsilon[W])_1 &:= \frac{1}{2} \sum_{m=1}^M \left(k_m^3 (\beta_{1,11}^m + \lambda \beta_{2,11}^m) \mathcal{A}_{k_m \epsilon} (\mathcal{A}_{k_m \epsilon} W_1)^2 + 2k_m^3 (\beta_{1,12}^m + \lambda \beta_{2,12}^m) \right. \\ &\quad \left. \mathcal{A}_{k_m \epsilon} \left((\mathcal{A}_{k_m \epsilon} W_1) (\mathcal{A}_{k_m \epsilon} W_2) \right) + k_m^3 (\beta_{1,22}^m + \lambda \beta_{2,22}^m) \mathcal{A}_{k_m \epsilon} (\mathcal{A}_{k_m \epsilon} W_2)^2 \right) \\ (\mathcal{Q}_\epsilon[W])_2 &:= \frac{\epsilon^2}{2} \sum_{m=1}^M \left(k_m^3 \beta_{2,11}^m \mathcal{A}_{k_m \epsilon} (\mathcal{A}_{k_m \epsilon} W_1)^2 + 2k_m^3 \beta_{2,12}^m \mathcal{A}_{k_m \epsilon} (\mathcal{A}_{k_m \epsilon} W_1) (\mathcal{A}_{k_m \epsilon} W_2) \right. \\ &\quad \left. + k_m^3 \beta_{2,22}^m \mathcal{A}_{k_m \epsilon} (\mathcal{A}_{k_m \epsilon} W_2)^2 \right) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{P}_\epsilon[W])_1 &:= \frac{1}{\epsilon^6} \sum_{m=1}^M k_m \left(\Psi_1^m (\epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_1, \epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_2) + \lambda \Psi_2^m (\epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_1, \epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_2) \right) \\ (\mathcal{P}_\epsilon[W])_2 &:= \frac{1}{\epsilon^4} \sum_{m=1}^M k_m \Psi_2^m (\epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_1, \epsilon^2 k_m \mathcal{A}_{k_m \epsilon} W_2). \end{aligned}$$

Notice that the asymptotic expansion of \mathcal{A}_η from (11) ensures that the formal limit equation (8) as $\epsilon \rightarrow 0$ can in fact be written as (9)+(10), where the ODE coefficients in (10) can be computed as

$$d_1 := \frac{(1 + \lambda^2)}{(a_1 + \lambda a_2) + \lambda(b_2 + \lambda b_1)}, \quad d_2 := \frac{(a_3 + \lambda^2 a_4 + \lambda a_5) + \lambda(\lambda b_5 + \lambda^2 b_3 + b_4)}{(a_1 + \lambda a_2) + \lambda(b_2 + \lambda b_1)} \quad (17)$$

with

$$\begin{aligned} a_1 &:= \sum_{m=1}^M \frac{k_m^4}{12} \alpha_{1,1}^m, a_2 := \sum_{m=1}^M \frac{k_m^4}{12} \alpha_{1,2}^m, a_3 := \sum_{m=1}^M \frac{k_m^3}{2} \beta_{1,11}^m, a_4 := \sum_{m=1}^M \frac{k_m^3}{2} \beta_{1,22}^m, a_5 := \sum_{m=1}^M k_m^3 \beta_{1,12}^m, \\ b_1 &:= \sum_{m=1}^M \frac{k_m^4}{12} \alpha_{2,2}^m, b_2 := \sum_{m=1}^M \frac{k_m^4}{12} \alpha_{2,1}^m, b_3 := \sum_{m=1}^M \frac{k_m^3}{2} \beta_{2,22}^m, b_4 := \sum_{m=1}^M \frac{k_m^3}{2} \beta_{2,11}^m, b_5 := \sum_{m=1}^M k_m^3 \beta_{2,12}^m. \end{aligned}$$

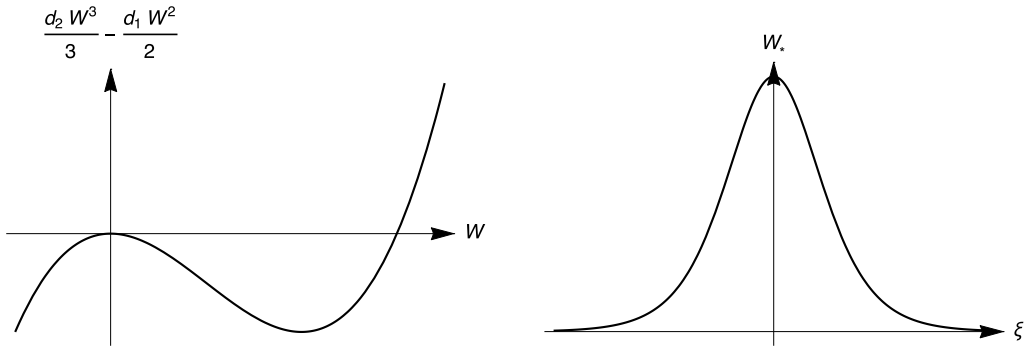


Figure 4: *Left panel:* Graph of the potential energy attached to the limit ODE; *Right panel:* The unique solution to the limit ODE in $\mathcal{L}_{\text{even}}^2(\mathbb{R})$, which corresponds to the region between the two zeros of $E[W] := \frac{d_2}{3}W^3 - \frac{d_1}{2}W^2$.

Eq.(10) describes a Hamiltonian system with potential energy $E[W] := \frac{1}{3}d_2W^3 - \frac{1}{2}d_1W^2$ and its coefficients are completely determined by the linear and quadratic terms of \tilde{F}_i^m . The unique homoclinic solution to (10) in $\mathbf{L}_{\text{even}}^2(\mathbb{R})$ is explicitly given by

$$W_*(\xi) = \frac{3d_1}{2d_2} \text{sech}^2\left(\frac{1}{2}\sqrt{d_1}\xi\right), \quad (18)$$

and a simple computation reveals that the corresponding orbit is naturally related to the shape parameters

$$p_1 := \max_{\xi \in \mathbb{R}} W_*(\xi) = \frac{3d_1}{2d_2} \quad p_2 := \max_{\xi \in \mathbb{R}} W'_*(\xi) = \sqrt{\frac{d_1^3}{3d_2^2}}.$$

In the following assumption we summarize the conditions, which guarantee a nontrivial limit profile W_0 in $\mathbf{L}_{\text{even}}^2(\mathbb{R})$.

Assumption 5 (Well-definedness of KdV waves) *The constants defined above satisfy*

1. $\sigma_0 > 0, c_2 \neq 0$;
2. $p_1 \neq 0, p_2^2 > 0$.

To facilitate further discussions we introduce two auxiliary functions

$$S_k(z) := 1 - \text{sinc}^2(kz/2), \quad s_{k,\epsilon}(z) := \frac{1 - \text{sinc}^2(k\epsilon z/2)}{\epsilon^2}, \quad s_{k,\epsilon}(z) = \frac{1}{\epsilon^2} S_k(\epsilon z), \quad (19)$$

for $z \in \mathbb{R}$, where $\text{sinc}(k\epsilon z/2)$ is the symbol function corresponding to $\mathcal{A}_{k\epsilon}$ in Fourier space. Since $\mathcal{A}_{k\epsilon}$ commute with one another, it's quite natural to consider the "determinant" of \mathcal{B}_ϵ defined by

$$\det \mathcal{B}_\epsilon := \mathcal{B}_{\epsilon,11}\mathcal{B}_{\epsilon,22} - \mathcal{B}_{\epsilon,12}\mathcal{B}_{\epsilon,21}.$$

Now suppose this operator is invertible on $\mathbf{L}^2(\mathbb{R})$. Then we define the matrix

$$\mathcal{B}_\epsilon^{-1} := (\det \mathcal{B}_\epsilon)^{-1} \begin{pmatrix} \mathcal{B}_{\epsilon,22} & -\mathcal{B}_{\epsilon,12} \\ -\mathcal{B}_{\epsilon,21} & \mathcal{B}_{\epsilon,11} \end{pmatrix},$$

and a straightforward calculation yields the following relation.

$$\mathcal{B}_\epsilon^{-1}\mathcal{B}_\epsilon = \mathcal{B}_\epsilon\mathcal{B}_\epsilon^{-1} = id_{(\mathbf{L}^2(\mathbb{R}))^2}.$$

Thus the problem of inverting \mathcal{B}_ϵ is reduced to that of inverting $\det \mathcal{B}_\epsilon$. We impose the following assumption to guarantee the invertibility of $\det \mathcal{B}_\epsilon$. Notice that the function $T(z)$ does not depend on ϵ . In section 3 we will check the required properties numerically.

Assumption 6 (Conditions for the inversion of \mathcal{B}_ϵ) *There exists a constant $\delta > 0$, such that*

$$T(z) := (2\sigma_0 - (c_1 + c_3)) \cdot \left(\sum_{m=1}^M k_m^2 \left((\sigma_0 - c_3)\alpha_{1,1}^m + (\sigma_0 - c_1)\alpha_{2,2}^m + 2c_2\alpha_{1,2}^m \right) S_{k_m}(z) \right) + \\ \left(\sum_{m=1}^M k_m^2 \alpha_{1,1}^m S_{k_m}(z) \right) \left(\sum_{m=1}^M k_m^2 \alpha_{2,2}^m S_{k_m}(z) \right) - \left(\sum_{m=1}^M k_m^2 \alpha_{1,2}^m S_{k_m}(z) \right)^2 \geq \delta_0 (\min\{|z|, 2\})^2$$

holds for all $z \in \mathbb{R}$.

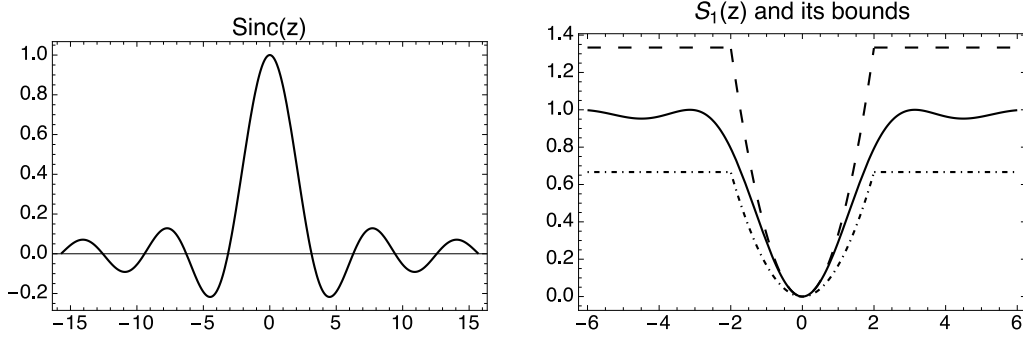


Figure 5: *Left panel:* Graph of the sinc function; *Right panel:* Lower bound $\frac{1}{6} \min\{|z|, 2\}^2$ (solid-dashed) and upper bound $\frac{1}{3} \min\{|z|, 2\}^2$ (dashed) for $S_1 = 1 - \text{sinc}^2$

Remark 7 The function T can be locally expanded as $T = \tau z^2 + O(z^4)$ near $z = 0$, where $\tau := \frac{1}{24} \sum_{m=1}^M k_m^4 \left((\sigma_0 - c_3) \alpha_{1,1}^m + (\sigma_0 - c_1) \alpha_{2,2}^m + 2c_2 \alpha_{1,2}^m \right)$. If $\tau > 0$, then the required condition is satisfied in the vicinity of $z = 0$ for any constant $0 < \delta < \tau_1$.

Lemma 8 For any $0 < \epsilon \leq 1$ the operator \mathcal{B}_ϵ is uniformly invertible on $(L^2(\mathbb{R}))^2$ under Assumptions 6. Moreover, we have the following estimate

$$\widehat{\det(\mathcal{B}_\epsilon)}(z) \geq \left| 2\sigma_0 - (c_1 + c_3) \right| + \frac{\delta}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2$$

for all $z \in \mathbb{R}$.

Proof: Let

$$T_1(z) := \sum_{m=1}^M k_m^2 (\alpha_{1,1}^m + \alpha_{2,2}^m) S_{k_m}(z).$$

For $m = 1, \dots, M$ there exists a constant $C_m > 0$, such that $|S_{k_m}(z)| \leq C_m (\min\{|z|, 2\})^2$. This implies the estimate

$$|T_1(z)| \leq \delta_1 (\min\{|z|, 2\})^2, \quad (20)$$

for all $z \in \mathbb{R}$ and a sufficiently large $\delta_1 > 0$. The Fourier symbol functions of the components of $\mathcal{B}_\epsilon^{\text{can}}$ are given by

$$\begin{aligned} \widehat{\mathcal{B}}_{\epsilon,11}^{\text{can}} &= \frac{1}{\epsilon^2} \left((\sigma_0 - c_1) + \epsilon^2 \left(1 + \sum_{m=1}^M k_m^2 \alpha_{1,1}^m s_{k_m,\epsilon} \right) \right), \quad \widehat{\mathcal{B}}_{\epsilon,22}^{\text{can}} = \frac{1}{\epsilon^2} \left((\sigma_0 - c_3) + \epsilon^2 \left(1 + \sum_{m=1}^M k_m^2 \alpha_{2,2}^m s_{k_m,\epsilon} \right) \right), \\ \widehat{\mathcal{B}}_{\epsilon,12}^{\text{can}} &= \widehat{\mathcal{B}}_{\epsilon,21}^{\text{can}} = \frac{1}{\epsilon^2} \left(-c_2 + \epsilon^2 \sum_{m=1}^M k_m^2 \alpha_{1,2}^m s_{k_m,\epsilon} \right). \end{aligned}$$

Now we compute the Fourier symbol function $\widehat{\det(\mathcal{B}_\epsilon)}$ of the determinant $\det(\mathcal{B}_\epsilon)$ by substituting

these expressions. This reads

$$\begin{aligned}
\widehat{\det(\mathcal{B}_\epsilon)} &= \widehat{\mathcal{T}_\epsilon} \cdot \widehat{\det(\mathcal{B}_\epsilon^{\text{can}})} = \epsilon^2 (\widehat{\mathcal{B}}_{\epsilon,11}^{\text{can}} \widehat{\mathcal{B}}_{\epsilon,22}^{\text{can}} - \widehat{\mathcal{B}}_{\epsilon,12}^{\text{can}} \widehat{\mathcal{B}}_{\epsilon,21}^{\text{can}}) \\
&= \frac{1}{\epsilon^2} \left(((\sigma_0 - c_1) + \epsilon^2 (1 + \sum_{m=1}^M k_m^2 \alpha_{1,1}^m s_{k_m,\epsilon}(z))) \cdot ((\sigma_0 - c_3) + \epsilon^2 (1 + \sum_{m=1}^M k_m^2 \alpha_{2,2}^m s_{k_m,\epsilon}(z))) - \right. \\
&\quad \left. (-c_2 + \epsilon^2 \sum_{m=1}^M k_m^2 \alpha_{1,2}^m s_{k_m,\epsilon}(z))^2 \right) \\
&= \frac{1}{\epsilon^2} \left((\sigma_0 - c_1) \cdot (\sigma_0 - c_3) - c_2^2 \right) + \\
&\quad \left(2\sigma_0 - (c_1 + c_3) \right) + \sum_{m=1}^M k_m^2 \left((\sigma_0 - c_3) \alpha_{1,1}^m + (\sigma_0 - c_1) \alpha_{2,2}^m + 2c_2 \alpha_{1,2}^m \right) s_{k_m,\epsilon}(z) + \\
&\quad \epsilon^2 \left(1 + \sum_{m=1}^M k_m^2 (\alpha_{1,1}^m + \alpha_{2,2}^m) s_{k_m,\epsilon}(z) \right. \\
&\quad \left. + \left(\sum_{m=1}^M k_m^2 \alpha_{1,1}^m s_{k_m,\epsilon}(z) \right) \left(\sum_{m=1}^M k_m^2 \alpha_{2,2}^m s_{k_m,\epsilon}(z) \right) - \left(\sum_{m=1}^M k_m^2 \alpha_{1,2}^m s_{k_m,\epsilon}(z) \right)^2 \right) \\
&\stackrel{(19), \text{Ass.6}}{=} \left(2\sigma_0 - (c_1 + c_3) + \epsilon^2 \right) + \frac{T(\epsilon z)}{(2\sigma_0 - (c_1 + c_3))\epsilon^2} + T_1(\epsilon z).
\end{aligned}$$

Without loss of generality we assume $2\sigma_0 - (c_1 + c_3) > 0$. (The other case is analogous.) By Assumption 6 and (20) we have

$$\frac{T(\epsilon z)}{(2\sigma_0 - (c_1 + c_3))\epsilon^2} + T_1(\epsilon z) \geq \left(\frac{\delta_0}{(2\sigma_0 - (c_1 + c_3))\epsilon^2} - \delta_1 \right) (\min\{|\epsilon z|, 2\})^2 \geq \frac{\delta}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2,$$

for sufficiently small $\epsilon > 0$. This implies

$$|\widehat{\det(\mathcal{B}_\epsilon)}| \geq \left| 2\sigma_0 - (c_1 + c_3) \right| + \frac{\delta}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2 \geq \left| 2\sigma_0 - (c_1 + c_3) \right| > 0,$$

for all $z \in \mathbb{R}$. In particular, $\widehat{\det(\mathcal{B}_\epsilon)}$ is bounded below by a positive constant independent of ϵ , so it is uniformly invertible. \square

Now we turn to the higher order terms Ψ_i^m . Roughly speaking, we would like them to be negligible as compared to the linear and quadratic terms. As the simplest case we have $\Psi_i^m \equiv 0$.

Assumption 9 (Regularity of higher order terms) *The remainder terms Ψ_i^m satisfy*

$$\Psi_i^m(0, 0) = 0$$

and

$$|\Psi_i^m(x_1, x_2) - \Psi_i^m(y_1, y_2)| \leq \gamma_i^m (x_1^2 + x_2^2 + y_1^2 + y_2^2) (|x_1 - y_1| + |x_2 - y_2|)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, $i = 1, 2$ and $m = 1, \dots, M$.

Having determined σ_0 and W_0 , we turn to the discussion of the equation for the corrector term V_ϵ . By substituting $W_\epsilon = W_0 + \epsilon^2 V_\epsilon$ into (7), we obtain

$$\mathcal{L}_\epsilon V_\epsilon := (\mathcal{B}_\epsilon - \mathcal{M}_\epsilon) V_\epsilon = \epsilon^2 (\mathcal{Q}_\epsilon[V_\epsilon] + \mathcal{N}_\epsilon[W_0; V_\epsilon]) + (R_\epsilon[W_0] + \mathcal{P}_\epsilon[W_0]), \quad (21)$$

where

$$R_\epsilon[W_0] := \frac{\mathcal{Q}_\epsilon[W_0] - \mathcal{B}_\epsilon W_0}{\epsilon^2}, \quad \mathcal{N}_\epsilon[W_0; V] := \frac{\mathcal{P}_\epsilon[W_0 + \epsilon^2 V] - \mathcal{P}_\epsilon[W_0]}{\epsilon^2}.$$

Introducing the abbreviations

$$\begin{aligned}\eta_{1,1}^m &:= k_m^3((\beta_{1,11}^m + \lambda\beta_{1,12}^m) + \lambda(\lambda\beta_{2,12}^m + \beta_{2,11}^m)), & \eta_{1,2}^m &:= k_m^3((\lambda\beta_{1,22}^m + \beta_{1,12}^m) + \lambda(\lambda\beta_{2,22}^m + \beta_{2,12}^m)), \\ \eta_{2,1}^m &:= k_m^3(\lambda\beta_{2,12}^m + \beta_{2,11}^m), & \eta_{2,2}^m &:= k_m^3(\lambda\beta_{2,22}^m + \beta_{2,12}^m)\end{aligned}$$

we write

$$\mathcal{M}_\epsilon = \begin{pmatrix} \sum_{m=1}^M \eta_{1,1}^m \mathcal{A}_{k_m\epsilon} \left((\mathcal{A}_{k_m\epsilon} W_*) \mathcal{A}_{k_m\epsilon} \right) & \sum_{m=1}^M \eta_{1,2}^m \mathcal{A}_{k_m\epsilon} \left((\mathcal{A}_{k_m\epsilon} W_*) \mathcal{A}_{k_m\epsilon} \right) \\ \epsilon^2 \sum_{m=1}^M \eta_{2,1}^m \mathcal{A}_{k_m\epsilon} \left((\mathcal{A}_{k_m\epsilon} W_*) \mathcal{A}_{k_m\epsilon} \right) & \epsilon^2 \sum_{m=1}^M \eta_{2,2}^m \mathcal{A}_{k_m\epsilon} \left((\mathcal{A}_{k_m\epsilon} W_*) \mathcal{A}_{k_m\epsilon} \right) \end{pmatrix}. \quad (22)$$

The uniform invertibility of \mathcal{L}_ϵ will be pivotal in the whole existence proof, for it allows us to write

$$V_\epsilon = F_\epsilon[V_\epsilon], \quad (23)$$

where $F_\epsilon[V_\epsilon] := \epsilon^2 \mathcal{L}_\epsilon^{-1}(\mathcal{Q}_\epsilon[V_\epsilon] + \mathcal{N}_\epsilon[W_0; V_\epsilon]) + \mathcal{L}_\epsilon^{-1}(R_\epsilon[W_0] + \mathcal{P}_\epsilon[W_0])$. With the contraction property of F_ϵ the Banach fixed point theorem will render us a solution.

2.3 Further properties of \mathcal{B}_ϵ , \mathcal{M}_ϵ and \mathcal{L}_ϵ

For later use we examine in this subsection the properties of the reformulated equation (21). As a preparation we define the cut-off operator

$$\Pi_\epsilon : \mathbb{L}^2(\mathbb{R}) \rightarrow \mathbb{L}^2(\mathbb{R})$$

by setting the corresponding symbol function in Fourier space

$$\pi_\epsilon(z) := \begin{cases} 1 & \text{for } |z| \leq \frac{2}{\epsilon}, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

In the next lemma we give some nice estimates of $\mathcal{B}_\epsilon^{-1}$ in combination with Π_ϵ , which remove the main difficulty in the inversion of \mathcal{L}_ϵ .

Lemma 10 *For any $0 < \epsilon \leq 1$ we have*

$$\|\Pi_\epsilon(\mathcal{B}_\epsilon^{-1})_{i1}G\|_{2,2} + \epsilon^{-2}\|(\text{id} - \Pi_\epsilon)(\mathcal{B}_\epsilon^{-1})_{i1}G\|_2 \leq D_4\|G\|_2, \quad \|(\mathcal{B}_\epsilon^{-1})_{i2}G\|_2 \leq D_4\|G\|_2,$$

for $G \in \mathbb{L}^2(\mathbb{R})$.

Proof: Note first that there exists for each $k > 0$ a constant $\kappa > 0$, such that $S_k(z) \leq \kappa(\min\{|z|, 2\})^2$ for all $z \in \mathbb{R}$. Applying this we have

$$\widehat{(\mathcal{B}_\epsilon^{-1})_{11}} = \widehat{\det(\mathcal{B}_\epsilon)}^{-1} \widehat{\mathcal{B}_{\epsilon,22}} \leq \frac{(\sigma_0 - c_3) + 1 + \sum_{m=1}^M k_m^2 |\alpha_{2,2}^m|}{(2\sigma_0 - (c_1 + c_3)) + \frac{\delta}{\epsilon^2}(\min\{|\epsilon z|, 2\})^2} \leq \begin{cases} \frac{C}{1+z^2} & \text{for } |z| \leq \frac{2}{\epsilon}, \\ C\epsilon^2 & \text{otherwise} \end{cases}$$

Similarly we have

$$\widehat{(\mathcal{B}_\epsilon^{-1})_{21}} = -\widehat{\det(\mathcal{B}_\epsilon)}^{-1} \widehat{\mathcal{B}_{\epsilon,21}} \leq \frac{|c_2| + \sum_{m=1}^M k_m^2 |\alpha_{1,2}^m|}{(2\sigma_0 - (c_1 + c_3)) + \frac{\delta}{\epsilon^2}(\min\{|\epsilon z|, 2\})^2} \leq \begin{cases} \frac{C}{1+z^2} & \text{for } |z| \leq \frac{2}{\epsilon}, \\ C\epsilon^2 & \text{otherwise.} \end{cases}.$$

This implies

$$\begin{aligned}\|\Pi_\epsilon(\mathcal{B}_\epsilon^{-1})_{i1}G\|_{2,2}^2 &\leq \int_{|z| \leq \frac{2}{\epsilon}} (1 + z^2 + z^4) \left(\widehat{(\mathcal{B}_\epsilon^{-1})_{i1}G} \right)^2 dz = \int_{|z| \leq \frac{2}{\epsilon}} (1 + z^2 + z^4) \widehat{(\mathcal{B}_\epsilon^{-1})_{i1}}^2 \widehat{G}^2 dz \\ &\leq C \int_{|z| \leq \frac{2}{\epsilon}} \frac{1 + z^2 + z^4}{1 + 2z^2 + z^4} \widehat{G}^2 dz \leq C\|G\|_2^2\end{aligned}$$

and

$$\begin{aligned} \|(\text{id} - \Pi_\epsilon)(\mathcal{B}_\epsilon^{-1})_{i1}G\|_{2,2}^2 &\leq \int_{|z| \geq \frac{2}{\epsilon}} \left(\widehat{(\mathcal{B}_\epsilon^{-1})_{i1}G} \right)^2 dz = \int_{|z| \geq \frac{2}{\epsilon}} \widehat{(\mathcal{B}_\epsilon^{-1})_{i1}}^2 \widehat{G}^2 dz \\ &\leq C\epsilon^4 \int_{|z| \geq \frac{2}{\epsilon}} \widehat{G}^2 dz \leq C\epsilon^4 \|G\|_2^2, \end{aligned}$$

for $G \in \mathbf{L}^2(\mathbb{R})$ and all $i = 1, 2$. Thus we have shown the first claimed estimate. For the second one we have

$$\widehat{(\mathcal{B}_\epsilon^{-1})_{12}} = \widehat{\det(\mathcal{B}_\epsilon)}^{-1} \widehat{\mathcal{B}_{\epsilon,12}} \leq \frac{\lambda + \left(\sum_{m=1}^M k_m^2 (|\alpha_{2,1}^m| + \lambda |\alpha_{2,2}^m|) \right) \frac{\kappa}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2}{(2\sigma_0 - (c_1 + c_3)) + \frac{\delta}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2} \leq C$$

and

$$\widehat{(\mathcal{B}_\epsilon^{-1})_{12}} = -\widehat{\det(\mathcal{B}_\epsilon)}^{-1} \widehat{\mathcal{B}_{\epsilon,12}} \leq \frac{\lambda + \left(\sum_{m=1}^M k_m^2 (|\alpha_{2,1}^m| + \lambda |\alpha_{2,2}^m|) \right) \frac{\kappa}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2}{(2\sigma_0 - (c_1 + c_3)) + \frac{\delta}{\epsilon^2} (\min\{|\epsilon z|, 2\})^2} \leq C$$

for all $z \in \mathbb{R}$. This implies

$$\|(\mathcal{B}_\epsilon^{-1})_{i2}G\|_{2,2}^2 \leq \int_{\mathbb{R}} \left(\widehat{(\mathcal{B}_\epsilon^{-1})_{i2}G} \right)^2 dz = \int_{\mathbb{R}} \widehat{(\mathcal{B}_\epsilon^{-1})_{i2}}^2 \widehat{G}^2 dz \leq C \int_{\mathbb{R}} \widehat{G}^2 dz \leq C \|G\|_2^2$$

for all $G \in \mathbf{L}^2(\mathbb{R})$. □

Lemma 11 *For any $0 < \epsilon \leq 1$ we have*

$$\|(\mathcal{M}_\epsilon G)_1\|_2 \leq D_5 \|G\|_2, \quad \|(\mathcal{M}_\epsilon G)_2\|_2 \leq \epsilon^2 D_5 \|G\|_2$$

for constants $D_4, D_5 > 0$ and for all $G \in \mathbf{L}^2(\mathbb{R})$ and $i = 1, 2$.

Proof: Lemma 3 yields

$$\|\mathcal{A}_{k_m\epsilon}(\mathcal{A}_{k_m\epsilon}W_*\mathcal{A}_{k_m\epsilon}G)\|_2 \leq \|\mathcal{A}_{k_m\epsilon}W_*\mathcal{A}_{k_m\epsilon}G\|_2 \leq \|\mathcal{A}_{k_m\epsilon}W_*\|_\infty \|\mathcal{A}_{k_m\epsilon}G\|_2 \leq \|W_*\|_\infty \|G\|_2$$

for all $G \in \mathbf{L}^2(\mathbb{R})$ and the claim follows from the definition of \mathcal{M}_ϵ , see (22). □

Above we discussed the properties of operators $\mathcal{B}_\epsilon^{-1}$ and \mathcal{M}_ϵ . Now we turn to the discussion of \mathcal{L}_ϵ and notice that its adjoint operator given by

$$\mathcal{L}_\epsilon^* V = \begin{pmatrix} \mathcal{B}_{\epsilon,11}V_1 + \mathcal{B}_{\epsilon,21}V_2 - \sum_m^M \mathcal{A}_{k_m\epsilon}(\mathcal{A}_{k_m\epsilon}W_*)(\eta_{1,1}^m \mathcal{A}_{k_m\epsilon}V_1 + \epsilon^2 \eta_{2,1}^m \mathcal{A}_{k\epsilon}V_2) \\ \mathcal{B}_{\epsilon,12}V_1 + \mathcal{B}_{\epsilon,22}V_2 - \sum_m^M \mathcal{A}_{k_m\epsilon}(\mathcal{A}_{k_m\epsilon}W_*)(\eta_{1,2}^m \mathcal{A}_{k_m\epsilon}V_1 + \epsilon^2 \eta_{2,2}^m \mathcal{A}_{k\epsilon}V_2) \end{pmatrix}$$

Passing to the limit $\epsilon \rightarrow 0$ in (21) we find $\mathcal{L}_0\phi := ((L_0\phi)_1, (L_0\phi)_2)$ with

$$\begin{aligned} (L_0\phi)_1 &= (\phi_1 + \lambda\phi_2) - (a_1 + \lambda a_2)\phi_1'' - (b_2 + \lambda b_1)\phi_2'' - 2W_*(a_3 + \lambda^2 a_4 + \lambda a_5)\phi_1 \\ &\quad - 2W_*(\lambda b_5 + \lambda^2 b_3 + b_4)\phi_2, \\ (L_0\phi)_2 &= -c_2\phi_1 + (\sigma_0 - c_3)\phi_2, \end{aligned}$$

see (16) and (22) for definitions of \mathcal{B}_ϵ and \mathcal{M}_ϵ , and similarly we get $\mathcal{L}_0^*\phi := ((L_0^*\phi)_1, (L_0^*\phi)_2)$ with

$$\begin{aligned} (\mathcal{L}_0^*\phi)_1 &= \phi_1 - (a_1 + \lambda a_2)\phi_1'' - 2W_*(a_3 + \lambda^2 a_4 + \lambda a_5)\phi_1 - c_2\phi_2, \\ (\mathcal{L}_0^*\phi)_2 &= \lambda\phi_1 - 2W_*(\lambda b_5 + \lambda^2 b_3 + b_4)\phi_1 - (b_2 + \lambda b_1)\phi_1'' + (\sigma_0 - c_3)\phi_2. \end{aligned}$$

The operators \mathcal{L}_ϵ^* and \mathcal{L}_0^* will play a role in the proof of the invertibility of \mathcal{L}_ϵ . The next lemma shows that \mathcal{L}_ϵ^* converges to \mathcal{L}_0^* for any sufficiently smooth test function.

Lemma 12 For any function ϕ with $\phi \in W^{4,2}(\mathbb{R})$ we have

$$\mathcal{L}_\epsilon^* \phi \xrightarrow{\epsilon \rightarrow 0} \mathcal{L}_0^* \phi$$

strongly in $L^2(\mathbb{R})$.

Proof: The operator \mathcal{L}_ϵ^* consists of linear combinations of $\frac{1-\mathcal{A}_{k\epsilon}^2}{\epsilon^2}$ and $\mathcal{A}_{k_m\epsilon}(\mathcal{A}_{k_m\epsilon}W_*)\mathcal{A}_{k_m\epsilon}$. Hence it suffices to show the L^2 -convergence of these operators for any test function $\phi \in W^{4,2}(\mathbb{R})$. Using Lemma 3 we have

$$\left\| \frac{1-\mathcal{A}_{k\epsilon}^2}{\epsilon^2} \phi - \left(-\frac{k^2}{12} \phi''\right) \right\|_2 \leq C\epsilon^2 \|\phi'''\|_2$$

and

$$\begin{aligned} \|\mathcal{A}_{k\epsilon}((\mathcal{A}_{k\epsilon}W_*)(\mathcal{A}_{k\epsilon}\phi)) - W_*\phi\|_2 &\leq \|\mathcal{A}_{k\epsilon}((\mathcal{A}_{k\epsilon}W_*)(\mathcal{A}_{k\epsilon}\phi)) - \mathcal{A}_{k\epsilon}((\mathcal{A}_{k\epsilon}W_*)\phi)\|_2 + \\ &\quad + \|\mathcal{A}_{k\epsilon}((\mathcal{A}_{k\epsilon}W_*)\phi) - \mathcal{A}_{k\epsilon}(W_*\phi)\|_2 + \|\mathcal{A}_{k\epsilon}(W_*\phi) - W_*\phi\|_2 \\ &\leq \|(\mathcal{A}_{k\epsilon}W_*)(\mathcal{A}_{k\epsilon}\phi - \phi)\|_2 + \|(\mathcal{A}_{k\epsilon}W_* - W_*)\phi\|_2 + \|\mathcal{A}_{k\epsilon}(W_*\phi) - W_*\phi\|_2 \\ &\leq \|W_*\|_\infty \|\mathcal{A}_{k\epsilon}\phi - \phi\|_2 + \|\phi\|_\infty \|\mathcal{A}_{k\epsilon}W_* - W_*\|_2 + \\ &\quad + \|\mathcal{A}_{k\epsilon}(W_*\phi) - W_*\phi\|_2 \\ &\leq Ck^2\epsilon^2(\|W_*\|_\infty \|\phi''\|_2 + \|\phi\|_\infty \|W_*''\|_2 + \|(W_*\phi)''\|_2). \end{aligned}$$

The assertion follows immediately. \square

Lemma 13 Under Assumption 5 the kernel of \mathcal{L}_0 in $(L^2(\mathbb{R}))^2$ is spanned by $W'_0 = (W'_*, \lambda W'_*)^T \in (L^2_{\text{odd}}(\mathbb{R}))^2$.

Proof: We first note that $\mathcal{L}_0\phi = 0$ is equivalent to $(L_0\phi)_1 = 0, (L_0\phi)_2 = 0$. The second equation gives $\phi_2 = \lambda\phi_1$. Substituting this back into the first equation, we obtain under Assumption 5

$$\phi_1'' = d_1\phi_1 - 2d_2W_*\phi_1, \quad (25)$$

where d_1, d_2 are given by (17). Eq.(25) can be viewed as the linearization of (7) and is solved by $W'_* \in L^2_{\text{odd}}(\mathbb{R})$. Now suppose ϕ_1 and ϕ_2 are two arbitrary solutions to (25) in $L^2(\mathbb{R})$. Then they also belong to the subspace $W^{1,2}(\mathbb{R})$. According to [23] we know

$$|\phi_i(\xi)| + |\phi'_i(\xi)| \xrightarrow{|\xi| \rightarrow \infty} 0$$

for $i = 1, 2$. Consider the Wronski determinant

$$\omega(\xi) := \begin{pmatrix} \phi_1(\xi) & \phi_2(\xi) \\ \phi'_1(\xi) & \phi'_2(\xi) \end{pmatrix}.$$

We know $\omega(\xi) \rightarrow 0$ for $|\xi| \rightarrow \infty$ and by using (25) we have $\omega'(\xi) = 0$. This implies $\omega \equiv 0$, which means that ϕ_1 and ϕ_2 are linearly dependent. Since W'_* solves (25), it spans its solution space. \square

The next lemma shows $R_\epsilon[W_0]$ and $\mathcal{P}_\epsilon[W_0]$ are uniformly bounded for small ϵ .

Lemma 14 There exists a constant $D_0 > 0$ independent of ϵ , such that

$$\|R_\epsilon[W_0] + \mathcal{P}_\epsilon[W_0]\|_2 \leq D_0,$$

for all $0 < \epsilon \leq 1$.

Proof: By formula (18) the function W_* is sufficiently regular in the sense that all its derivatives belong to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Lemma 3 implies

$$\begin{aligned} \|\mathcal{A}_{k\epsilon}(\mathcal{A}_{k\epsilon}W_*)^2 - W_*^2\|_2 &\leq \|\mathcal{A}_{k\epsilon}(\mathcal{A}_{k\epsilon}W_*)^2 - \mathcal{A}_{k\epsilon}W_*^2\|_2 + \|\mathcal{A}_{k\epsilon}W_*^2 - W_*^2\|_2 \\ &\leq (\|\mathcal{A}_{k\epsilon}W_*\|_\infty + \|W_*\|_\infty)\|\mathcal{A}_{k\epsilon}W_* - W_*\|_2 + Ck^2\epsilon^2 \leq C\epsilon^2 \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{1 - \mathcal{A}_{k\epsilon}^2}{\epsilon^2} W_* - \left(-\frac{k^2}{12} W_*''\right) \right\|_2 &= \frac{1}{\epsilon^2} \|(\mathcal{A}_{k\epsilon} + \text{id})(\mathcal{A}_{k\epsilon}W_* - W_* - \frac{k^2\epsilon^2}{24} W_*'') + \frac{k^2\epsilon^2}{24} (\mathcal{A}_{k\epsilon}W_*'' - W_*'')\|_2 \\ &\leq \frac{1}{\epsilon^2} \|(\mathcal{A}_{k\epsilon} + \text{id})(\mathcal{A}_{k\epsilon}W_* - W_* - \frac{k^2}{24} W_*'')\|_2 + \frac{k^2}{24} \|\mathcal{A}_{k\epsilon}W_*'' - W_*''\|_2 \\ &\leq \frac{2}{\epsilon^2} \|\mathcal{A}_{k\epsilon}W_* - W_* - \frac{k^2\epsilon^2}{24} W_*''\|_2 + \frac{k^2}{24} \|\mathcal{A}_{k\epsilon}W_*'' - W_*''\|_2 \\ &\leq C\epsilon^2 \|W_*'''\|_2, \end{aligned}$$

where C is a constant, which only depends on k . Using these estimates we obtain

$$\|(R_\epsilon[W_0])_1\|_2 = \left\| \frac{1}{\epsilon^2} \left(-(1 + \lambda^2)W_* + h_1W_*'' + h_2W_*^2 + \epsilon^2\Delta R_\epsilon \right) \right\|_2 = \|\Delta R_\epsilon\|_2 \leq D_1.$$

Applying again Lemma 3 we have

$$\|\mathcal{A}_{k\epsilon}(\mathcal{A}_{k\epsilon}W_*)^2\|_2 \leq \|\mathcal{A}_{k\epsilon}W_*\|_\infty \|\mathcal{A}_{k\epsilon}W_*\|_2 \leq \|W_*\|_\infty \|W_*\|_2, \quad \|\mathcal{A}_{k\epsilon}^2 W_*\|_2 \leq \|W_*\|_2,$$

which implies immediately

$$\|(R_\epsilon[W_0])_2\|_2 \leq D_2,$$

for $0 < \epsilon \leq 1$ and a constant $D_2 > 0$. Now we estimate $\mathcal{P}_\epsilon[W_0]$. Assumption (4) implies

$$\begin{aligned} \|\Psi_i^m(\epsilon^2 k_m \mathcal{A}_{k_m\epsilon} W_*, \lambda \epsilon^2 k_m \mathcal{A}_{k_m\epsilon} W_*)\|_2 &\leq \gamma_i^m k_m^3 (1 + \lambda^2) (1 + |\lambda|) \epsilon^6 \|(\mathcal{A}_{k_m\epsilon} W_*)^2 (\mathcal{A}_{k_m\epsilon} W_*)\|_2 \\ &\leq \gamma_i^m k_m^3 (1 + \lambda^2) (1 + |\lambda|) \epsilon^6 \|(\mathcal{A}_{k_m\epsilon} W_*)\|_\infty^2 \|\mathcal{A}_{k_m\epsilon} W_*\|_2 \\ &\leq \gamma_i^m k_m^3 (1 + \lambda^2) (1 + |\lambda|) \epsilon^6 \|W_*\|_\infty^2 \|W_*\|_2 \end{aligned}$$

for all $i = 1, 2$ and $m = 1, \dots, M$. Since $0 < \epsilon \leq 1$, we have

$$\|\mathcal{P}_\epsilon[W_0]\|_2 \leq D_3,$$

and the assertion follows by setting $D_0 := D_1 + D_2 + D_3$. \square

2.4 Inverting the Operator \mathcal{L}_ϵ

Our approach requires to invert \mathcal{L}_ϵ , but since \mathcal{L}_0 has a nontrivial kernel, see Lemma 13, we cannot expect that $\mathcal{L}_\epsilon^{-1}$ exists on $(L^2(\mathbb{R}))^2$. We can, however, prove the existence of $\mathcal{L}_\epsilon^{-1}$ on the space of even functions.

Lemma 15 *The subspace $(L_{\text{even}}^2(\mathbb{R}))^2$ is invariant in $(L^2(\mathbb{R}))^2$ under the linear mapping \mathcal{L}_ϵ .*

Proof: According to the definition all the components of \mathcal{L}_ϵ are linearly spanned by $\mathcal{A}_{k_m\epsilon}^2$ and $(\mathcal{A}_{k_m\epsilon}(\mathcal{A}_{k_m\epsilon}W_*))\mathcal{A}_{k_m\epsilon}$. We know $\mathcal{A}_{k_m\epsilon}$ respects the even-odd parity and W_* is an even function. The assertion follows. \square

The following auxiliary lemma will be used in the inversion of \mathcal{L}_ϵ .

Lemma 16 Suppose $u \in \mathbf{L}^2(\mathbb{R})$ and $|\int_{\mathbb{R}} u \phi''| \leq C \|\phi\|_2$ for a constant $C > 0$ and any smooth function ϕ with compact support in \mathbb{R} . Then $u \in \mathbf{W}^{2,2}(\mathbb{R})$.

Proof: See [23] for the proof. □

Now we are in a position to state and prove the main theorem of this section.

Theorem 17 (Invertibility of \mathcal{L}_ϵ) There exists a constant $\epsilon_* > 0$ such that for any $\epsilon \in (0, \epsilon_*)$ the operator \mathcal{L}_ϵ is invertible on $\mathbf{L}_{\text{even}}^2(\mathbb{R})$. More precisely there exists a constant C_* which does not depend on ϵ such that

$$\|\mathcal{L}_\epsilon^{-1} G\|_2 \leq C_* \|G\|_2$$

for all $\epsilon \in (0, \epsilon_*)$ and all $G \in (\mathbf{L}_{\text{even}}^2(\mathbb{R}))^2$.

Proof: Preliminaries: We will show that there is a constant $c_* > 0$ such that

$$\|\mathcal{L}_\epsilon V\|_2 \geq c_* \|V\|_2 \quad (26)$$

for all $V \in (\mathbf{L}_{\text{even}}^2(\mathbb{R}))^2$ and all sufficiently small ϵ . This implies that \mathcal{L}_ϵ is injective and has closed image. The same holds for the symmetric operator $\mathcal{T}_\epsilon^{-1} \circ \mathcal{L}_\epsilon$ under Assumption 9. Thus it is also surjective and we obtain in turn the surjectivity of \mathcal{L}_ϵ . Due to Inequality (26) \mathcal{L}_ϵ has a continuous inverse $\mathcal{L}_\epsilon^{-1}$.

Antithesis: We suppose that such a constant c_* does not exist. Then there exists a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \epsilon_n = 0$ as well as sequences $(V_n)_{n \in \mathbb{N}}$ and $(G_n)_{n \in \mathbb{N}} \subset (\mathbf{L}_{\text{even}}^2(\mathbb{R}))^2$ such that

$$\mathcal{L}_{\epsilon_n} V_n = G_n, \quad \|V_n\|_2 = 1, \quad \|G_n\| \xrightarrow{n \rightarrow \infty} 0.$$

In view of the invertibility of \mathcal{B}_ϵ we have

$$V_n = \mathcal{B}_{\epsilon_n}^{-1} (\mathcal{M}_{\epsilon_n} V_n + G_n) = \begin{pmatrix} (\mathcal{B}_{\epsilon_n}^{-1})_{11} (\mathcal{M}_{\epsilon_n} V_n + G_n)_1 + (\mathcal{B}_{\epsilon_n}^{-1})_{12} (\mathcal{M}_{\epsilon_n} V_n + G_n)_2 \\ (\mathcal{B}_{\epsilon_n}^{-1})_{21} (\mathcal{M}_{\epsilon_n} V_n + G_n)_1 + (\mathcal{B}_{\epsilon_n}^{-1})_{22} (\mathcal{M}_{\epsilon_n} V_n + G_n)_2 \end{pmatrix},$$

where $(\mathcal{M}_{\epsilon_n} V_n + G_n)_1, (\mathcal{M}_{\epsilon_n} V_n + G_n)_2$ are components of $(\mathcal{M}_{\epsilon_n} V_n + G_n)$. Lemmas 10 and 11 give

$$\|(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_2 \leq \|(\mathcal{M}_{\epsilon_n} V_n)_1\|_2 + \|G_n\|_2 \leq D_5 \|V_n\|_2 + \|G_n\|_2 = D_5 + \|G_n\|_2$$

and

$$\|(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 \leq \|(\mathcal{M}_{\epsilon_n} V_n)_2\|_2 + \|G_n\|_2 \leq D_5 \epsilon^2 \|V_n\|_2 + \|G_n\|_2 = D_5 \epsilon^2 + \|G_n\|_2.$$

Weak convergence to 0: Since the sequence $(V_n)_{n \in \mathbb{N}}$ is bounded, there exists a $V_\infty \in (\mathbf{L}_{\text{even}}^2(\mathbb{R}))^2$ such that

$$V_n \xrightarrow{n \rightarrow \infty} V_\infty \text{ weakly in } (\mathbf{L}^2(\mathbb{R}))^2.$$

Due to Lemma 12 we have

$$\langle V_\infty, \mathcal{L}_0^* \phi \rangle = \lim_{n \rightarrow \infty} \langle V_n, \mathcal{L}_{\epsilon_n}^* \phi \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{L}_{\epsilon_n} V_n, \phi \rangle = \lim_{n \rightarrow \infty} \langle G_n, \phi \rangle = 0,$$

for any sufficiently smooth function ϕ . Let $\phi_1, (V_\infty)_1$ denote the first components of ϕ, V_∞ and $\phi_2, (V_\infty)_2$ the second. Putting $\phi_1 = 0$ we obtain

$$\langle -c_2 (V_\infty)_1 + (\sigma_0 - c_3) (V_\infty)_2, \phi_2 \rangle = 0,$$

for any sufficiently smooth function ϕ_2 . Since ϕ_2 is arbitrary, we have

$$(V_\infty)_2 = \frac{c_2}{\sigma_0 - c_3}(V_\infty)_1 = \lambda(V_\infty)_1.$$

Substituting this back into the equation $\langle V_\infty, \mathcal{L}_0^* \phi \rangle = 0$ and letting $\phi_2 = 0$ we obtain

$$\begin{aligned} & \left\langle (V_\infty)_1, (1 + \lambda^2)\phi_1 - \left((a_1 + \lambda a_2) + \lambda(b_2 + \lambda b_1) \right) \phi_1'' \right. \\ & \quad \left. - 2W_* \left((a_3 + \lambda^2 a_4 + \lambda a_5) + \lambda(\lambda b_5 + \lambda^2 b_3 + b_4) \right) \phi_1 \right\rangle = 0, \end{aligned}$$

for any sufficiently smooth function ϕ_1 and this implies

$$\left| \int_{\mathbb{R}} (V_\infty)_1 \phi_1'' \right| \leq C \|\phi_1\|_2.$$

Lemma 16 implies that $(V_\infty)_1$ belongs to $W^{2,2}(\mathbb{R})$. Thus V_∞ belongs to $(W^{2,2}(\mathbb{R}))^2$. This allows us to apply \mathcal{L}_0 and

$$\langle \mathcal{L}_0 V_\infty, \phi \rangle = \langle V_\infty, \mathcal{L}_0^* \phi \rangle = 0$$

holds for any sufficiently smooth function ϕ , which implies that the function V_∞ is contained in the kernel of \mathcal{L}_0 . Since it is even, Lemma 13 can be applied and we have

$$V_\infty = 0.$$

Further notations: We choose a constant $K > 0$ such that

$$\sup_{|\zeta| \geq K - k_{\max}} W_*(\zeta) \leq \frac{1}{4D_4 \sum_{m=1}^M (|\eta_{1,1}^m| + |\eta_{1,2}^m|)}$$

and denote by χ_K the characteristic function of the interval $I_K := [-K, +K]$. In order to estimate V_n we break it up in the following way

$$V_n = V_n^{(1)} + V_n^{(2)} + V_n^{(3)},$$

where the $V_n^{(i)}$'s are given by

$$V_n^{(1)} = \begin{pmatrix} \chi_K \Pi_{\epsilon_n} V_{n,1} \\ \chi_K \Pi_{\epsilon_n} V_{n,2} \end{pmatrix}, \quad V_n^{(2)} = \begin{pmatrix} (1 - \chi_K) \Pi_{\epsilon_n} V_{n,1} \\ (1 - \chi_K) \Pi_{\epsilon_n} V_{n,2} \end{pmatrix}, \quad V_n^{(3)} = \begin{pmatrix} (\text{id} - \Pi_{\epsilon_n}) V_{n,1} \\ (\text{id} - \Pi_{\epsilon_n}) V_{n,2} \end{pmatrix}$$

and $V_{n,1}, V_{n,2}$ are the components of V_n .

Strong convergence of $V_n^{(1)}$ and $V_n^{(3)}$: We first show the convergence of $V_n^{(3)}$. Lemmas 10 and 11 yield

$$\begin{aligned} \|V_n^{(3)}\|_2 & \leq \|(\text{id} - \Pi_{\epsilon_n})(\mathcal{B}_{\epsilon_n}^{-1})_{11}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_2 + \|(\text{id} - \Pi_{\epsilon_n})(\mathcal{B}_{\epsilon_n}^{-1})_{21}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_2 + \\ & \quad \|(\mathcal{B}_{\epsilon_n}^{-1})_{12}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 + \|(\mathcal{B}_{\epsilon_n}^{-1})_{22}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 \\ & \leq 2\epsilon_n D_4 \|(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_2 + 2D_5 \|(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 \\ & \leq 2\epsilon_n D_4 (D_5 + \|G_n\|_2) + 2D_5 (D_5 \epsilon^2 + \|G_n\|_2), \end{aligned}$$

so the sequence $V_n^{(3)}$ converges strongly to 0. To decompose $V_n^{(1)}$ we define

$$U_n^{(1)} = \begin{pmatrix} \chi_K \Pi_{\epsilon_n} (\mathcal{B}_{\epsilon_n}^{-1})_{11}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1 \\ \chi_K \Pi_{\epsilon_n} (\mathcal{B}_{\epsilon_n}^{-1})_{21}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1 \end{pmatrix}, \quad U_n^{(2)} = \begin{pmatrix} \chi_K \Pi_{\epsilon_n} (\mathcal{B}_{\epsilon_n}^{-1})_{12}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2 \\ \chi_K \Pi_{\epsilon_n} (\mathcal{B}_{\epsilon_n}^{-1})_{22}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2 \end{pmatrix}.$$

Then we have automatically

$$V_n^{(1)} = U_n^{(1)} + U_n^{(2)}.$$

Lemmas 10 and 11 yield

$$\begin{aligned} \|U_n^{(1)}\|_{1,2,I_K} &\leq \|\Pi_{\epsilon_n}(\mathcal{B}_{\epsilon_n}^{-1})_{11}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_{1,2} + \|\Pi_{\epsilon_n}(\mathcal{B}_{\epsilon_n}^{-1})_{21}(\mathcal{M}_{\epsilon_n} V_n + G_n)_1\|_{1,2} \\ &\leq 2D_4 \|\mathcal{M}_{\epsilon_n} V_n + G_n\|_2 \leq 2D_4 D_5 \end{aligned}$$

and

$$\begin{aligned} \|U_n^{(2)}\|_2 &\leq \|\Pi_{\epsilon_n}(\mathcal{B}_{\epsilon_n}^{-1})_{12}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 + \|\Pi_{\epsilon_n}(\mathcal{B}_{\epsilon_n}^{-1})_{22}(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 \\ &\leq 2D_4 \|(\mathcal{M}_{\epsilon_n} V_n + G_n)_2\|_2 \leq 2D_4 (D_5 \epsilon_n^2 + \|G_n\|_2). \end{aligned}$$

So we have

$$U_n^{(2)} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbf{L}^2(I_K))^2.$$

We already know

$$V_n \xrightarrow{n \rightarrow \infty} V_\infty = 0 \text{ weakly in } (\mathbf{L}^2(I_K))^2, \quad V_n^{(3)} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbf{L}^2(I_K))^2,$$

the function $V_n^{(2)}$ is supported in $\mathbb{R} \setminus I_K$ and $(W^{1,2}(I_K))^2$ is compactly embedded into $(\mathbf{L}^2(I_K))^2$. From the identity

$$V_n = U_n^{(1)} + U_n^{(2)} + V_n^{(2)} + V_n^{(3)}$$

we conclude

$$U_n^{(1)} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbf{L}^2(I_K))^2.$$

In summary we find

$$V_n^{(1)} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbf{L}^2(I_K))^2$$

and this implies

$$V_n^{(1)} \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbf{L}^2(\mathbb{R}))^2,$$

since $V_n^{(1)}$ is supported I_K .

Upper bounds for V_n : Let $(V_n^{(2)})_i$ be the i -th component of $V_n^{(2)}$. Then we have

$$\begin{aligned} \|\mathcal{A}_{k\epsilon_n}(\mathcal{A}_{k\epsilon_n} W_* \mathcal{A}_{k\epsilon_n} (V_n^{(2)})_i)\|_2 &\leq \|\mathcal{A}_{k\epsilon_n} W_* \mathcal{A}_{k\epsilon_n} (V_n^{(2)})_i\|_2 \leq \left(\sup_{|\zeta| \geq K - k_{\max}} W_*(\zeta) \right) \|\mathcal{A}_{k\epsilon_n} (V_n^{(2)})_i\|_2 \\ &\leq \sup_{|\zeta| \geq K - k_{\max}} W_*(\zeta). \end{aligned}$$

Together with Lemmas 10 and 11 this implies

$$\begin{aligned} \|\mathcal{B}_{\epsilon_n}^{-1} \mathcal{M}_{\epsilon_n} V_n^{(2)}\|_2 &\leq \|(\mathcal{B}_{\epsilon_n}^{-1})_{11}(\mathcal{M}_{\epsilon_n} V_n^{(2)})_1\|_2 + \|(\mathcal{B}_{\epsilon_n}^{-1})_{21}(\mathcal{M}_{\epsilon_n} V_n^{(2)})_1\|_2 + \\ &\quad \|(\mathcal{B}_{\epsilon_n}^{-1})_{12}(\mathcal{M}_{\epsilon_n} V_n^{(2)})_2\|_2 + \|(\mathcal{B}_{\epsilon_n}^{-1})_{22}(\mathcal{M}_{\epsilon_n} V_n^{(2)})_2\|_2 \\ &\leq 2(1 + \epsilon) D_4 \|(\mathcal{M}_{\epsilon_n} V_n^{(2)})_1\|_2 + D_4 \|(\mathcal{M}_{\epsilon_n} V_n^{(2)})_2\|_2 \\ &\leq 2(1 + \epsilon) D_4 \left(\sum_{m=1}^M (|\eta_{1,1}^m| + |\eta_{1,2}^m|) \right) \left(\sup_{|\zeta| \geq K - k_{\max}} W_*(\zeta) \right) + \epsilon^2 D_4 \\ &\leq \frac{1}{2} (1 + \epsilon) + \epsilon^2 D_4. \end{aligned}$$

Passing to the limit $n \rightarrow \infty$ we get

$$\limsup_{n \rightarrow \infty} \|\mathcal{B}_{\epsilon_n}^{-1} \mathcal{M}_{\epsilon_n} V_n^{(2)}\|_2 \leq \frac{1}{2}.$$

Derivation of the contradiction: The inequality

$$\|V_n\|_2 \leq \|\mathcal{B}_{\epsilon_n}^{-1} \mathcal{M}_{\epsilon_n} V_n^{(1)}\|_2 + \|\mathcal{B}_{\epsilon_n}^{-1} \mathcal{M}_{\epsilon_n} V_n^{(2)}\|_2 + \|\mathcal{B}_{\epsilon_n}^{-1} \mathcal{M}_{\epsilon_n} V_n^{(3)}\|_2$$

implies

$$\limsup_{n \rightarrow \infty} \|V_n\|_2 \leq \frac{1}{2}.$$

This contradicts the normalization condition $\|V_n\|_2 = 1$ and we conclude that the antithesis is in fact false. \square

2.5 Fixed-point Argument

To conclude the proof it remains to verify the conditions of the Banach fixed-point theorem.

Theorem 18 (existence and uniqueness of the solution to (23)) *Under Assumptions 5, 6 and 9 there exist constants $D > 0$ and $\epsilon_* > 0$, such that for any $\epsilon < \epsilon_*$ the operator \mathcal{F}_ϵ has a unique fixed point in the ball $B_D = \{V : (\mathbb{L}_{\text{even}}^2(\mathbb{R}))^2 : \|V\|_2 \leq D\}$.*

Proof: We demonstrate that the operator \mathcal{F}_ϵ is a contraction in a sufficiently large ball B_D for any sufficiently small ϵ , thus satisfying all the conditions of the Banach fixed point theorem. We don't fix our D at first.

Estimates for the quadratic terms: For arbitrary $V_1, V_2 \in B_D$ Assumption 9 gives us

$$\begin{aligned} \|\mathcal{A}_{k_m \epsilon}(\mathcal{A}_{k_m \epsilon} V_{1,i})^2 - \mathcal{A}_{k_m \epsilon}(\mathcal{A}_{k_m \epsilon} V_{2,i})^2\|_2 &\leq \|(\mathcal{A}_{k_m \epsilon} V_{1,i})^2 - (\mathcal{A}_{k_m \epsilon} V_{2,i})^2\|_2 \\ &\leq \left(\|\mathcal{A}_{k_m \epsilon} V_{1,i}\|_\infty + \|\mathcal{A}_{k_m \epsilon} V_{2,i}\|_\infty \right) \|\mathcal{A}_{k_m \epsilon} V_{1,i} - \mathcal{A}_{k_m \epsilon} V_{2,i}\|_2 \\ &\leq k_m^{-1/2} \epsilon^{-1/2} (\|V_{1,i}\|_2 + \|V_{2,i}\|_2) \|V_{1,i} - V_{2,i}\|_2 \\ &\leq 2D k_m^{-1/2} \epsilon^{-1/2} \|V_1 - V_2\|_2 \end{aligned}$$

and

$$\begin{aligned} &\|\mathcal{A}_{k_m \epsilon} \left((\mathcal{A}_{k_m \epsilon} V_{1,1})(\mathcal{A}_{k_m \epsilon} V_{1,2}) \right) - \mathcal{A}_{k_m \epsilon} \left((\mathcal{A}_{k_m \epsilon} V_{2,1})(\mathcal{A}_{k_m \epsilon} V_{2,2}) \right)\|_2 \\ &\leq \|(\mathcal{A}_{k_m \epsilon} V_{1,1})(\mathcal{A}_{k_m \epsilon} V_{1,2}) - (\mathcal{A}_{k_m \epsilon} V_{1,1})(\mathcal{A}_{k_m \epsilon} V_{2,2})\|_2 \\ &\quad + \|(\mathcal{A}_{k_m \epsilon} V_{1,1})(\mathcal{A}_{k_m \epsilon} V_{2,2}) - (\mathcal{A}_{k_m \epsilon} V_{2,1})(\mathcal{A}_{k_m \epsilon} V_{2,2})\|_2 \\ &\leq \|\mathcal{A}_{k_m \epsilon} V_{1,1}\|_\infty \|\mathcal{A}_{k_m \epsilon} V_{1,2} - \mathcal{A}_{k_m \epsilon} V_{2,2}\|_2 + \|\mathcal{A}_{k_m \epsilon} V_{2,2}\|_\infty \|\mathcal{A}_{k_m \epsilon} V_{1,1} - \mathcal{A}_{k_m \epsilon} V_{2,1}\|_2 \\ &\leq k_m^{-1/2} \epsilon^{-1/2} \left(\|V_{1,1}\|_2 \|V_{1,2} - V_{2,2}\|_2 + \|V_{2,2}\|_2 \|V_{1,1} - V_{2,1}\|_2 \right) \\ &\leq D k_m^{-1/2} \epsilon^{-1/2} \|V_1 - V_2\|_2 \end{aligned}$$

Applying this to \mathcal{Q}_ϵ we have

$$\|\epsilon^2 \mathcal{Q}_\epsilon[V_1] - \epsilon^2 \mathcal{Q}_\epsilon[V_2]\|_2 \leq CD \epsilon^{3/2} \|V_1 - V_2\|_2$$

for $V_1, V_2 \in B_D$. It follows

$$\|\epsilon^2 \mathcal{Q}_\epsilon[V]\|_2 \leq CD \epsilon^{3/2} \|V\|_2 \leq CD^2 \epsilon^{3/2}$$

for any $V \in B_D$. We emphasize that C is a constant, which does not depend on D .

Estimates for the higher order terms: Assumption 9 gives us

$$\begin{aligned}
& \left\| \Psi_i^m \left(\epsilon^2 k_m \mathcal{A}_{k_m \epsilon}(W_* + \epsilon^2 V_{1,1}), \epsilon^2 k_m \mathcal{A}_{k_m \epsilon}(\lambda W_* + \epsilon^2 V_{1,2}) \right) - \right. \\
& \quad \left. \Psi_i^m \left(\epsilon^2 k_m \mathcal{A}_{k_m \epsilon}(W_* + \epsilon^2 V_{2,1}), \epsilon^2 k_m \mathcal{A}_{k_m \epsilon}(\lambda W_* + \epsilon^2 V_{2,2}) \right) \right\|_2 \\
& \leq \gamma_i^m k_m^{-1} \epsilon^5 (\|W_* + \epsilon^2 V_{1,1}\|_2^2 + \|W_* + \epsilon^2 V_{1,2}\|_2^2 + \|W_* + \epsilon^2 V_{2,1}\|_2^2 + \|W_* + \epsilon^2 V_{2,2}\|_2^2) \\
& \quad \epsilon^2 (\|V_{1,1} - V_{2,1}\|_2 + \|V_{1,2} - V_{2,2}\|_2) \\
& \leq 4\gamma_i^m \epsilon^7 (2\|W_*\|_2^2 + \epsilon^4 D^2) \|V_1 - V_2\|_2 \leq \epsilon^7 C(C + \epsilon^4 D^2) \|V_1 - V_2\|_2
\end{aligned}$$

for $V_1, V_2 \in B_D$. This implies

$$\|\epsilon^2 \mathcal{N}_\epsilon[W_0; V_1] - \epsilon^2 \mathcal{N}_\epsilon[W_0; V_2]\|_2 = \|\mathcal{P}_\epsilon[W_0 + V_1] - \mathcal{P}_\epsilon[W_0 + V_2]\|_2 \leq \epsilon C(C + \epsilon^4 D^2) \|V_1 - V_2\|_2$$

for $V_1, V_2 \in B_D$. It follows

$$\|\epsilon^2 \mathcal{N}_\epsilon[W_0; V]\|_2 \leq \epsilon C(C + \epsilon^4 D^2) \|V\|_2 \leq \epsilon C D(C + \epsilon^4 D^2)$$

for all $V \in B_D$.

Concluding arguments: Lemma 14 gives

$$\|R_\epsilon[W_0] + \mathcal{P}_\epsilon[W_0]\|_2 \leq C.$$

This implies

$$\|\mathcal{F}_\epsilon[V]\|_2 \leq C + C D^2 \epsilon^{3/2} + \epsilon C D(C + \epsilon^4 D^2)$$

for any $V \in B_D$. With the estimates above we have

$$\|\mathcal{F}_\epsilon[V_1] - \mathcal{F}_\epsilon[V_2]\|_2 \leq \left(C D \epsilon^{3/2} + \epsilon C(C + \epsilon^4 D^2) \right) \|V_1 - V_2\|_2$$

for all $V_1, V_2 \in B_D$. Now we choose D to be $2C$. We find a constant ϵ_* , such that $C + C D^2 \epsilon_*^{3/2} + \epsilon_* C D(C + \epsilon_*^4 D^2) \leq D$ and $(C D \epsilon_*^{3/2} + \epsilon_* C(C + \epsilon_*^4 D^2)) < 1$ hold. The Banach fixed point theorem gives the existence and uniqueness of V_ϵ for all $0 < \epsilon \leq \epsilon_*$. \square

3 Applications to different lattices

We consider three different lattices in this section. The existence of a solitary solution as given by Theorem 1 involves the verification of Assumptions 4, 5, 6 and 9. Notice that since all the relevant effective potentials are smooth at the point $(0, 0)$, Assumptions 4 and 9 are automatically satisfied. Thus only two assumptions remain to be verified, which will be done numerically. For an intuitive picture we shall also give the behavior W_0 and λ with respect to angle α .

We first remind that (14) gives two possible candidates for σ_0 , but in our approach we always chose the larger solution, see (15). The reason is that although the other solution branch might give rise to a well-defined KdV wave in the limit $\epsilon \rightarrow 0$, we cannot expect the corresponding operator \mathcal{B}_ϵ to be invertible. To see this, we observe that \mathcal{B}_ϵ is invertible if and only if σ_ϵ does not belong to the spectrum of the operator $\mathcal{J}_\epsilon := \sigma_\epsilon \text{id} - \epsilon^2 \mathcal{B}_\epsilon^{\text{can}}$. Since the Fourier symbol of the latter is given by

$$\hat{\mathcal{J}}_{\epsilon, ij}(z) = \sum_{m=1}^M k_m^2 \alpha_{i,j}^m \text{sinc}^2(k_m \epsilon z / 2),$$

we conclude that

$$\text{spec}\mathcal{J}_\epsilon = \cup_{i=1,2}\{\mu_i(\epsilon z) : \epsilon z \in \mathbb{R}\},$$

where the functions μ_1, μ_2 are completely determined by the coefficients k_m and $\alpha_{i,j}^m$, see Figure 6 for an illustration. Moreover, by construction we have $\sigma_0 \in \{\mu_1(0), \mu_2(0)\}$, so in the case of $\sigma_0 = \max\{\mu_1(0), \mu_2(0)\}$ Assumption 6 guarantees that

$$\max_{i=\{1,2\}, z \in \mathbb{R}} \{\mu_i(z)\} = \sigma_0 < \sigma_\epsilon$$

and hence the desired invertibility of \mathcal{B}_ϵ . For $\sigma_0 = \min\{\mu_1(0), \mu_2(0)\}$, however, Assumption 6 can generically not be satisfied. In this case we expect that KdV-types waves still exist, but exhibit oscillatory tails due to resonances with the continuous spectrum. A proof of this fact would involve sophisticated arguments lying beyond the scope of the present paper, but we refer to [11, 12] for similar rigorous results on certain generalizations of one-dimensional FPU chains.

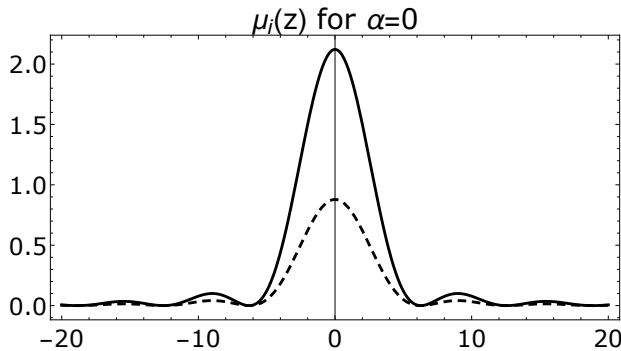


Figure 6: Auxiliary functions μ_1 and μ_2 for the square lattice with $V_1 = V_2 = \frac{1}{2}(r-1)^2$ and $r_* = 0.8047$ in solid and dashed lines, respectively.

3.1 Square lattice

As the first example we continue the discussion of the square lattice. We emphasize here as a complementary remark that the introduction of the diagonal springs leads to the non-linearity necessary for the existence of solitary waves. Without diagonal springs there would be no resistance against the shear forces and the mechanical structure would become unstable. For the expressions of the effective potentials and k_m see section 1.1.

Test Assumption 5: The purpose of Assumption 5 is to ensure the existence of KdV traveling waves in the formal limit $\epsilon \rightarrow 0$. This is actually a minimal condition for the existence of a solitary wave in its neighborhood. In Figure 7 we see that σ_0 is always positive and it appears as oscillations around a constant and $p_1 \neq 0$ and p_2 can be defined for all $\alpha \in [0, \frac{\pi}{4}]$. Due to axial and rotational symmetry of the square lattice we have a KdV wave for all $\alpha \in [0, 2\pi)$.

Test Assumption 6: We list in Figure 8 the numerically computed functions $T(z)$ for $\alpha = 0, \frac{\pi}{12}, \frac{\pi}{6}$ and $\frac{\pi}{4}$. By Remark 7 the local behavior of T around $z = 0$ complies with Assumption 5. These graphs give numerical evidence for Assumption 6.

The limit velocity profiles: As an illustration of the α -dependence of the wave, we give in Figure 9 the two components of the limit velocity profiles for the same set of values of α as in Figure 8. In $\alpha = \frac{\pi}{4}$ we observe the coincidence of the two components of W_0 , which is in consistence with the axial symmetry of the lattice with respect to the diagonal direction.

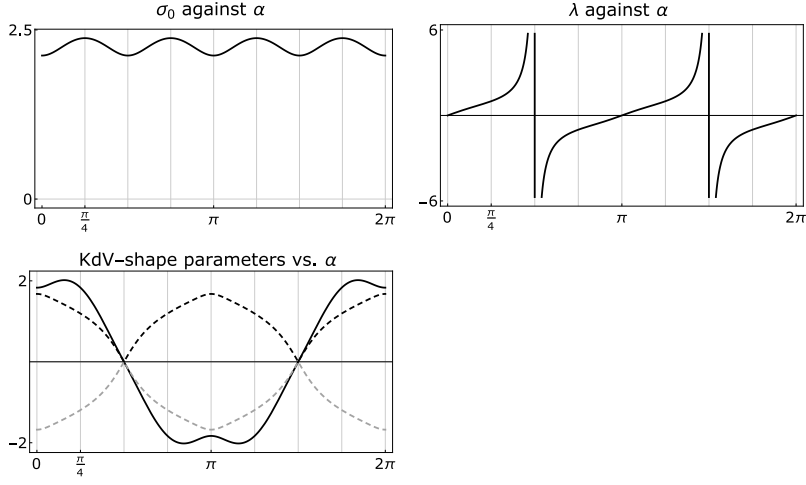


Figure 7: *Upper row:* The behavior of σ_0 , λ with respect to α ; *Bottom left:* KdV-shape parameters p_1, p_2 . The positive and negative roots of p_2^2 are given in dashed and grey lines respectively.

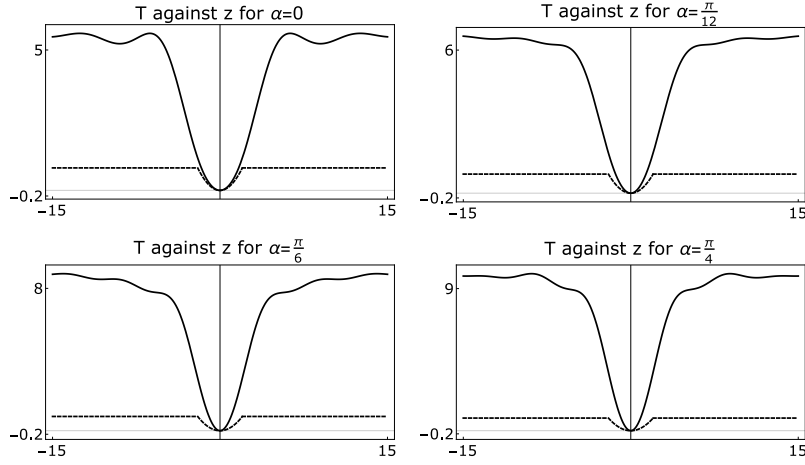


Figure 8: $T(z)$ (solid) and $g(z) = 0.3 \cdot (\min\{z, 2\})^2$ (dashed). Assumption 6 requires $T(z) \geq g(z)$ for all $z \in \mathbb{R}$.

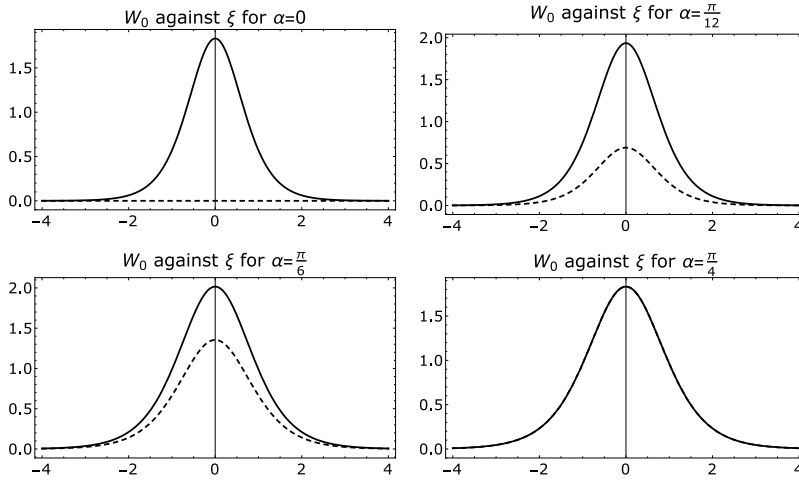


Figure 9: Examples of the KdV profiles for selected values of α . *Solid lines:* first component of W_0 ; *Dashed lines:* second component of W_0 .

3.2 Diamond lattice

The second example, which is called diamond lattice and illustrated in Figure 10, arises by taking out the springs on one diagonal of the square lattice and then rotating by 45 degrees. Thus for each particle three pairs of forces come into question and the number of symmetries is considerably reduced as compared with the square lattice; we still have the two reflection symmetries and but no more invariance under rotations by 45 degrees.

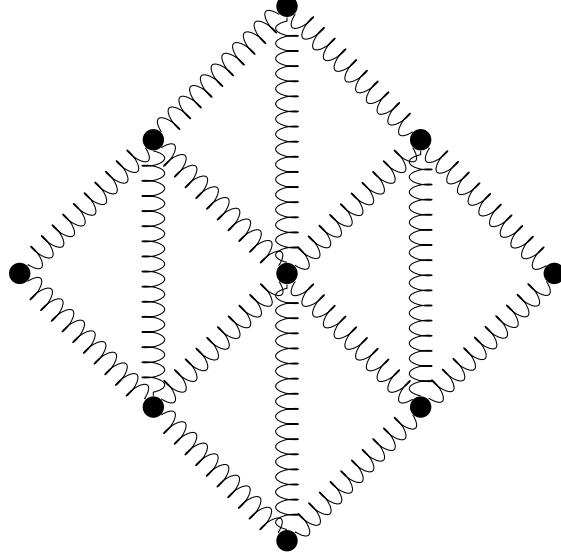


Figure 10: Diamond lattice.

Effective k_m : Similarly as before, the k_m are obtained by projection onto the axes of symmetry. These are here $(0, 1)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. The results are given as follows.

$$k_1 = \sin(\alpha), \quad k_2 = \frac{1}{\sqrt{2}} \cos(\alpha) + \frac{1}{\sqrt{2}} \sin(\alpha), \quad k_3 = \frac{1}{\sqrt{2}} \cos(\alpha) - \frac{1}{\sqrt{2}} \sin(\alpha).$$

Effective potentials: Since potentials only depend on distance, it suffices to find out the distances for a horizontal displacement x_1 and a vertical displacement x_2 . This is easily done by applying Pythagoras' theorem. Substituting the expressions of distances into ϕ , we obtain the following effective potentials.

$$\begin{aligned} \phi_1(x_1, x_2) &= V(\sqrt{x_1^2 + (x_2 + r_*)^2}), & \phi_2(x_1, x_2) &= V(\sqrt{(x_1 + \frac{1}{\sqrt{2}}r_*)^2 + (x_2 + \frac{1}{\sqrt{2}}r_*)^2}), \\ \phi_3(x_1, x_2) &= V(\sqrt{(x_1 + \frac{1}{\sqrt{2}}r_*)^2 + (x_2 - \frac{1}{\sqrt{2}}r_*)^2}). \end{aligned}$$

Test Assumption 5: In Figure 11 we see that σ_0 is always positive and it exhibits considerable changes with α instead of the oscillatory behavior in the square lattice. Due to axial symmetry of the lattice it suffices to consider $\alpha \in [0, \frac{\pi}{2}]$.

In the graph of KdV-shape parameters it is easily seen that $p_1 \neq 0$ and p_2 can be defined for all $\alpha \in (0, \frac{\pi}{4}]$ except for a critical point. Due to axial symmetry of the square lattice we have a KdV wave for all α .

Test Assumption 6: In Figure 12 the numerically computed function $T(z)$ is given for $\alpha = \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$. We don't give the graph for $\alpha = 0$, since the KdV-shape parameters explode for this

value. Similarly as before, Assumption 5 is fulfilled locally around $z = 0$ and Figure 12 gives then numerical evidence for its global validity.

The limit velocity profiles: We observe in Figure 13 a clear flip of the velocity profiles, when α turns from $\frac{\pi}{12}$ to $\frac{\pi}{6}$. This means that the wave is compressive for small values of α and above a certain value it becomes expansive. The threshold for this change is exactly the singularity between 0 and $\frac{\pi}{2}$ as seen in the (p_1, p_2) - α graph of Figure 11. In $\alpha = \frac{\pi}{2}$ we observe the vanishing of the first component of W_0 , which hints at a similar result as proven by Friesecke and Matthies [3], namely the existence of a unidirectional and longitudinal KdV-like solitary wave, whose first component completely vanishes.

3.3 Triangle lattice

For an illustration of the third example we refer to the right panel of Figure 1. As a comparison we keep the physical potential and the lattice parameter r_* to be the same as in the last two examples.

Effective k_m : Similarly as before, the k_m are obtained by projecting κ onto the axes of symmetry. These are here $(1, 0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The results are given as follows.

$$k_1 = \cos(\alpha), \quad k_2 = \frac{1}{2} \cos(\alpha) + \frac{\sqrt{3}}{2} \sin(\alpha), \quad k_3 = \frac{1}{2} \cos(\alpha) - \frac{\sqrt{3}}{2} \sin(\alpha).$$

Effective potentials: Applying Pythagoras' theorem and substituting the expressions of distances into ϕ , we obtain the following effective potentials.

$$\begin{aligned} \phi_1(x_1, x_2) &= V(\sqrt{(x_1 + r_*)^2 + x_2^2}), \quad \phi_2(x_1, x_2) = V(\sqrt{(x_1 + \frac{1}{2}r_*)^2 + (x_2 + \frac{\sqrt{3}}{2}r_*)^2}), \\ \phi_3(x_1, x_2) &= V(\sqrt{(x_1 + \frac{1}{2}r_*)^2 + (x_2 - \frac{\sqrt{3}}{2}r_*)^2}). \end{aligned}$$

Test Assumption 5: By comparison of Figures 7, 11 and 14 it appears that the more the symmetries the respective lattice possesses, the less dramatically σ_0 will change with the angle α . As shown by Figure 14, the square σ_0 of the sound speed is constant for all α . We have, however, no accurate formulation and no rigorous proof of this observation.

Test Assumption 6: In Figure 15 the numerically computed function $T(z)$ is given for $\alpha = 0, \frac{\pi}{18}, \frac{\pi}{9}, \frac{\pi}{6}$, which provides numerical evidence for Assumption 6.

The limit velocity profiles: In Figure 16 we give KdV velocity profiles for the same set of values for α as in Figure 12. It is clearly seen that we always obtain expansion waves. Similarly as before, we find unidirectional and longitudinal KdV-like solitary waves along the symmetry axes $\alpha = 0, \frac{\pi}{2}$. In all cases, the global uniqueness of solutions remains undetermined.

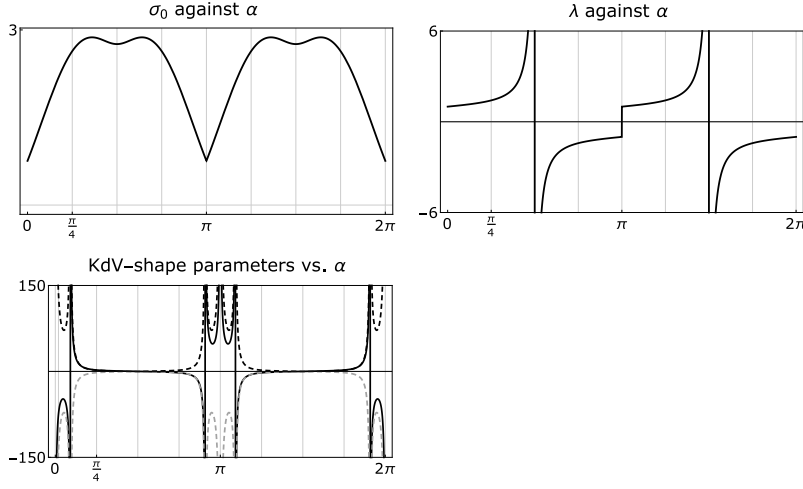


Figure 11: The plots from Figure 7 for the diamond lattice. Notice that in the graph of λ we find jumps at multiples of π , which is consistent with the fact that the lattice is symmetric with respect to the horizontal direction. For $\alpha = 0$ no KdV wave exists due to the singularity.

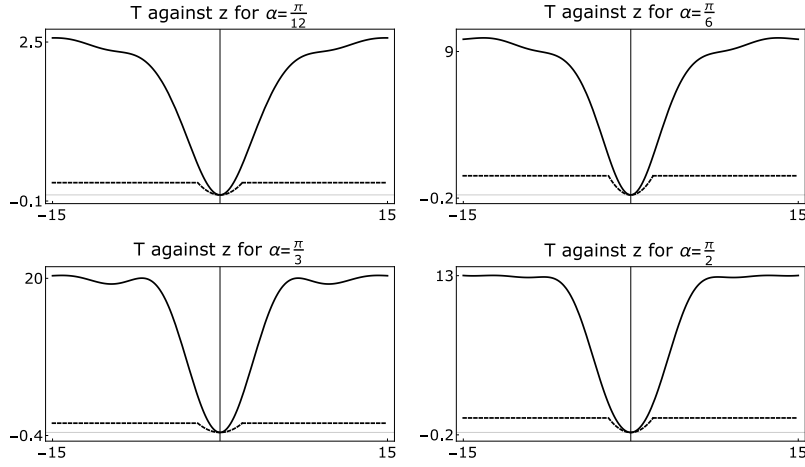


Figure 12: The plots from Figure 8 for the diamond lattice. *Solid lines:* $T(z)$; *Dashed lines:* $g = 1.3 \cdot (\min\{z, 2\})^2$.

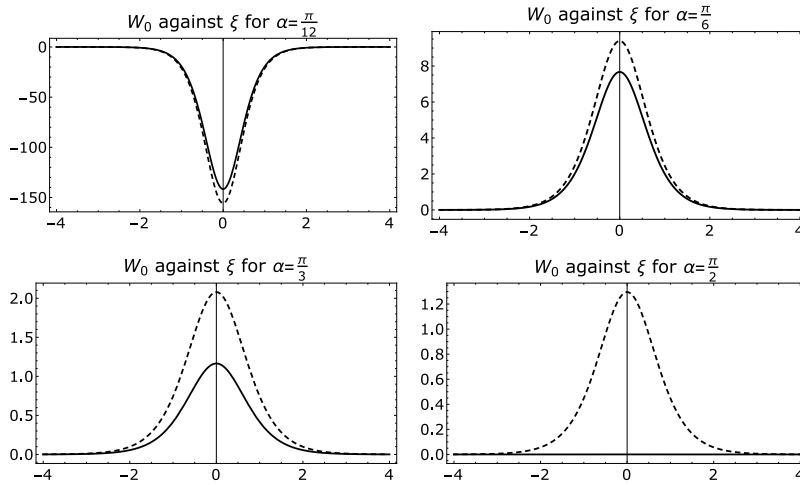


Figure 13: The plots from Figure 9 for the diamond lattice.

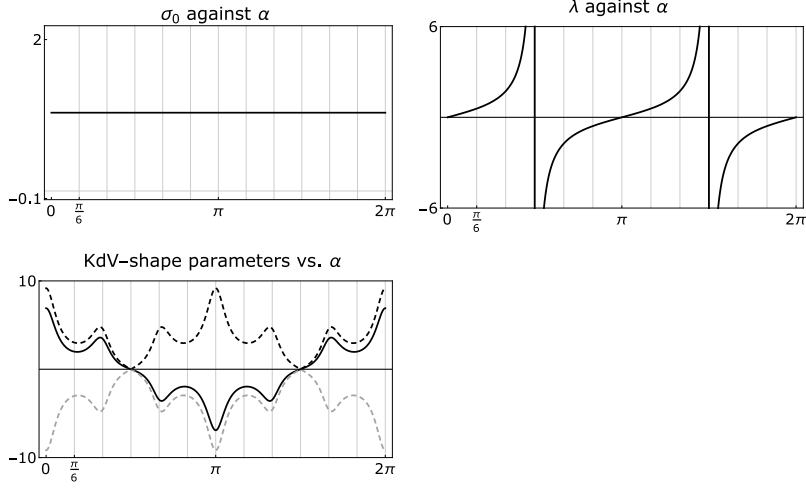


Figure 14: The plots from Figure 7 for the triangle lattice.

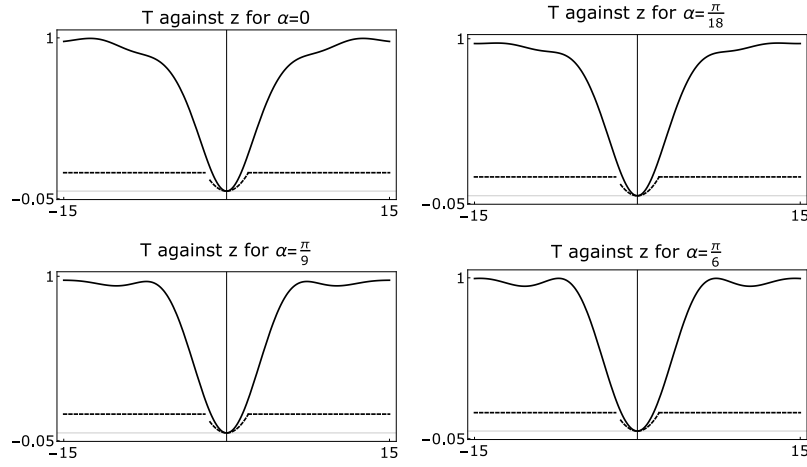


Figure 15: The plots from Figure 8 for the triangle lattice. *Solid lines:* $T(z)$; *Dashed lines:* $g = 0.3 \cdot (\min\{z, 2\})^2$.

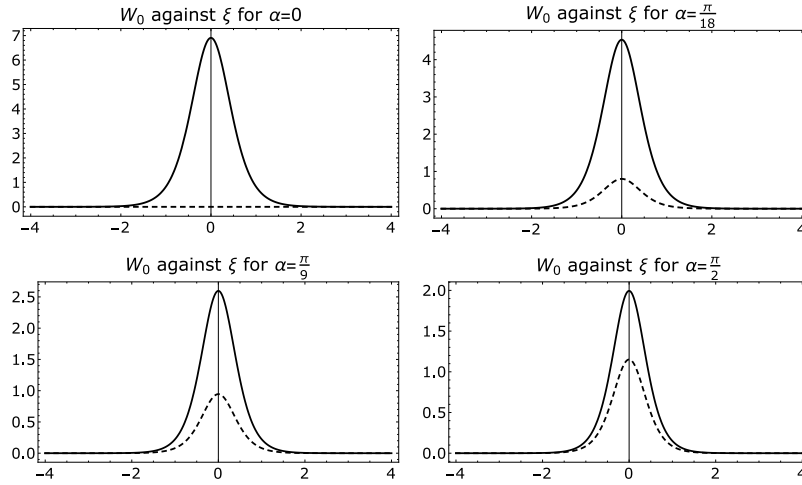


Figure 16: The plots from Figure 9 for the triangle lattice.

Acknowledgments

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